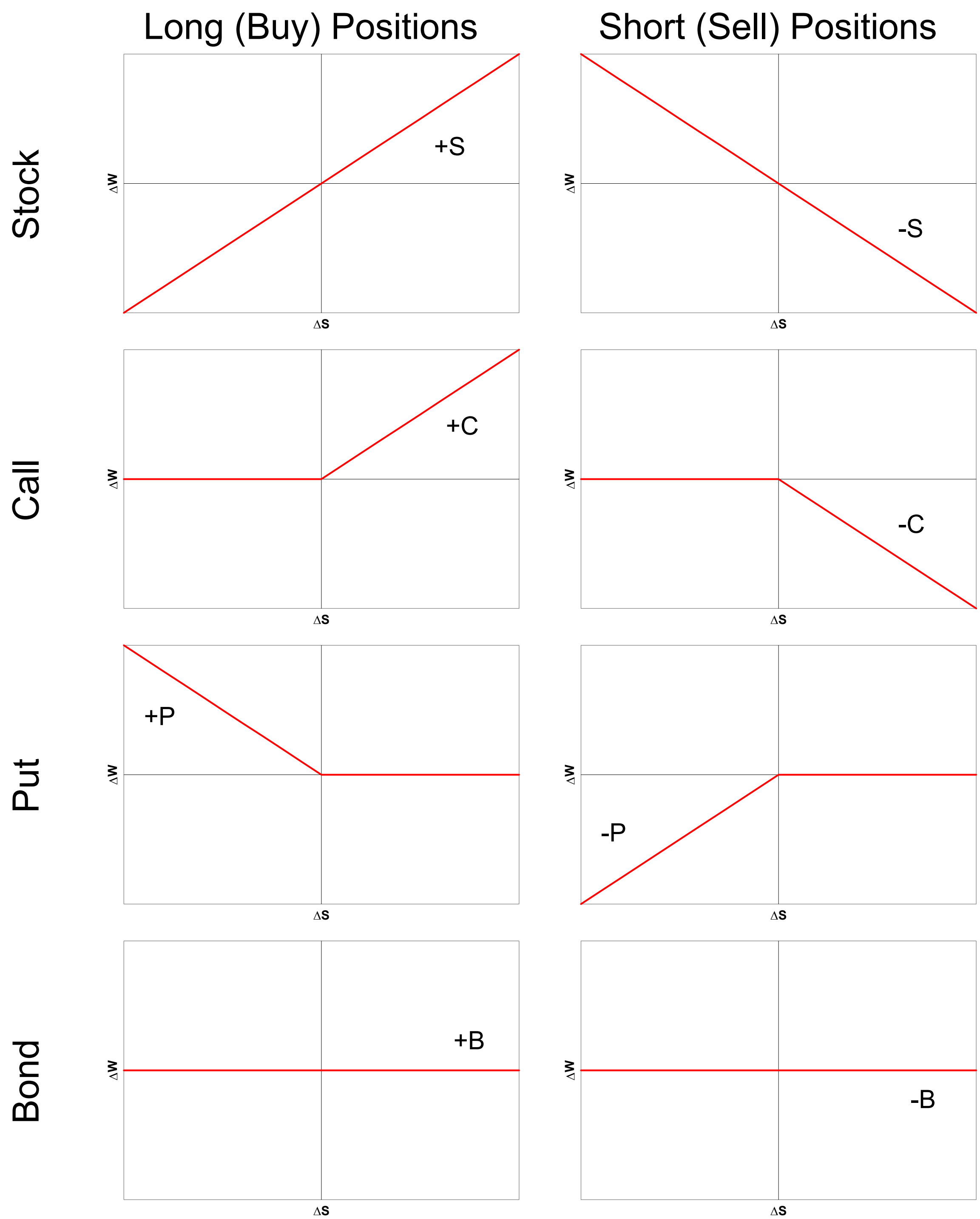
Option Fundamentals

Payoff Diagrams – These are the basic building blocks of financial engineering. They represent the ***payoffs*** or ***terminal values*** of various investment choices. We shall assume that the maturity value of the bond is $X, the exercise price.

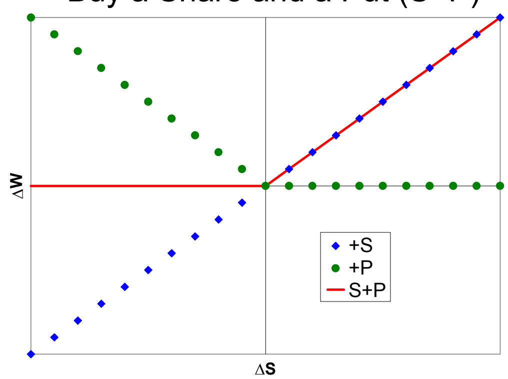


Payoff Positions – The algebra of the payoffs is shown in the table below. The exercise price of the options will be defined as $X and we shall assume that the maturity value of the bond is $X

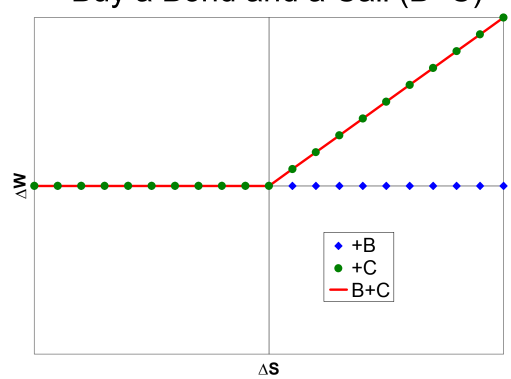
Stock Price	S	C	P	B
S > X	S	S – X	nil	X
S < X	S	nil	X – S	X

Financial Alchemy – What happens when we combine some of these investments

Buy a Share and a Put (S+P)



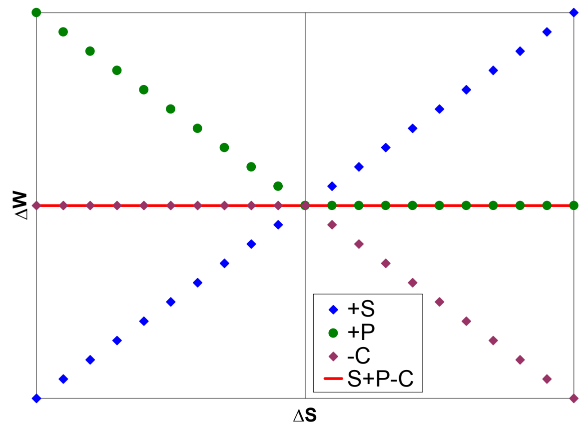
Buy a Bond and a Call (B+C)



Notice that these have the same pattern of payoffs, given the change in the underlying stock price. The payoffs are:

	S + P		B + C	
S > X	S:	S	B:	X
	P:	0	C:	S – X
	Total	S	Total	S
S < X	S:	S	B:	X
	P:	X – S	C:	0
	Total	X	Total	X

With options, it is possible to create a risk free payoff:



	S + P – C	
S > X	S:	S
	P:	0
	-C:	– (S – X)
	Total	X
S < X	S:	S
	P:	X – S
	-C:	0
	Total	X

Note that this creates a risk free payoff, which is always \$X. Therefore, we can create a risk-free security by buying a stock, buying a put option with an exercise price of \$X and selling (or writing) a call option with an exercise price of \$X.

Binomial Pricing

Let's assume that we have a stock currently selling for \$40 and there is an at-the-money option that expires in 6 months, ½ a year. The risk free rate is 4.04% or 2% over the six-month time horizon. We shall assume that the outcomes are binary, that is that the stock can either rise 25% or fall 20%. Note that these two rates imply terminal values that are the multiplicative inverse of each other, (1+25%) = 1/(1-20%). Thus the terminal values for the stock can be either \$32 or \$50. We shall examine two strategies to allow us to work toward pricing the option.

Strategy #1 – Buy a Call

	P_6 = 40(1-20%) = \$32	P_6 = 40(1+25%) = \$50
Call Value	nil	50 – 40 = \$10

Strategy #2 – Buy 5/9s of a share and take out a risk-free loan with a repayment of \$17.777… The loan is (\$17.777…)/(1.02) = 17.42919…

	P_6 = 40(1-20%) = \$32	P_6 = 40(1+25%) = \$50
Stock Value	\$32 x (5/9) = \$17.777…	\$50 x (5/9) = \$27.777…
Loan Repayment	(\$17.777…)	(\$17.777…)
Total	nil	\$10

Notice that these two strategies have identical payoffs, meaning that the two strategies must have the same value:

$$C = (5/9)(40) - 17.42919\ldots = \$4.793\ldots$$

Where did the 5/9s come from?

This is the “Delta” (Δ) of the option. It is also known as the hedge ratio.

$$\Delta = \frac{\text{Spread of Options Values}}{\text{Spread of Share Values}} = \frac{10.00 - 0}{50.00 - 32.00} = \frac{10.00}{18.00} = \frac{5}{9}$$

Where did the $17.777… come from?

It is the difference between the in-the-money share payoff (5/9 x $50 = 27.777…) and the option payoff ($10.00). The amount borrowed is the present value, discounted at the risk-free rate of interest.

Based on what we have seen with creating risk free payoffs, the option value that we obtained, $4.793…, is a fair value. Any other price would create arbitrage opportunities.

Risk Neutral Valuation

Let’s take another approach, and assume that investors do not care about risk. Since we demonstrated that we could create risk-free payoffs using options, this approach is not much of a stretch.

Since investors are indifferent about risk, the return expected on the stock over the next six months is 2%. Let’s compute the risk-neutral probability that the stock will rise (P_U):

$$\begin{aligned}
&E(r) = 2\% = (P_U)(25\%) + (1 - P_U)(-20\%) \\
&\Rightarrow 2\% = (25\%)(P_U) + (20\%)(P_U) - (20\%) \\
&\Rightarrow 22\% = (45\%)(P_U) \\
&\Rightarrow (P_U) = (22\%)/(45\%) = 0.48888...
\end{aligned}$$

Let’s apply this probability to the option payoff or terminal value:

Probability	Terminal Value	Cross-Product
0.4888…	$10.00	4.888…
0.5111…	0.00	0.00
	Total	4.888…

This is the expected value of the option at the maturity. The present value of this is $4.888…/1.02 = $4.793…, the same value that we had before.

The general formula to get this risk-neutral probability of an upside price movement is:

$P_U = \dfrac{r - D}{U - D}$, where r is the risk free return to the maturity of the option.

Put Option Valuation

Strategy #1 – Buy a Put

	P_6 = 40(1-20%) = $32	P_6 = 40(1+25%) = $50
Put Value	40 – 32 = $8	nil

Strategy #2 – Sell 4/9s of a share and lend $21.786… as a risk-free loan with a repayment of $22.222…

	P_6 = 40(1-20%) = $32	P_6 = 40(1+25%) = $50
Stock Value	$32 x (-4/9) = ($14.22…)	$50 x (-4/9) = ($22.22…)
Loan Repayment	($22.222…)	($22.222…)
Total	$8	Nil

Notice that these two strategies have identical payoffs, meaning that the two strategies must have the same value:

P = (-4/9)(40) + 21.786… = $4.0087…

Using Risk Neutral Valuation:

Probability	Terminal Value	Cross-Product
0.4888…	$0.00	0.00
0.5111…	8.00	4.0888…
	Total	4.0888…

This is the expected value of the option at the maturity. The present value of this is $4.0888…/1.02 = $4.0087…, which is the same value.

Put-Call Parity

Recall that:

$$S + P = B + C$$
$$\text{and } B = Xe^{-rT}$$
$$S + P = Xe^{-rT} + C$$
$$P = C - S + Xe^{-rT}$$

In our example:

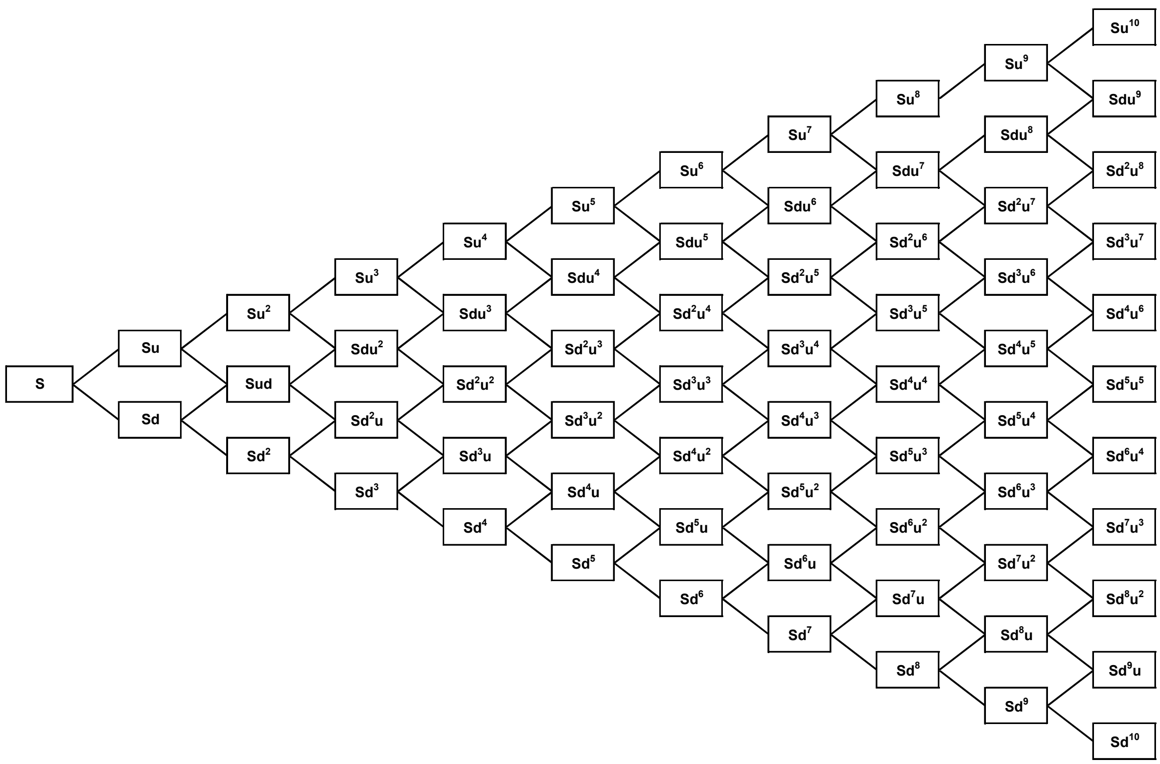
$$P = \$4.793\ldots - 30.00 + (30.00)^{-0.02}$$
$$= \$4.0087\ldots$$

General Form of Put-Call Parity

$$C - P = S - Xe^{-rT}$$

Generalizing Binomial Option Pricing

We assumed that there were only two terminal values of the stock in the future, S(1+U) and S(1+D). To make this more realistic, we split the time preiods into smaller fractions of the year and adjust the u = (1 + U) and d = (1 + D) movements accordingly. As an intermediate step the ***binomial lattice*** would look like this:



As the time intervals or steps get smaller and the ending points become continuous, we have the familiar situations of the binomial approaching the normal. The difference here is that the probabilities are not symmetrical and the distribution of ending prices is actually the lognormal.

To determine the appropriate values of U and D (or u and d) we need the following relationships:

$$1 + U = u = e^{\sigma T}$$
$$1 + D = d = 1/u$$

Applying this to our example:

$$1.25 = e^{\sigma\sqrt{0.5}}$$

$$\ln(1.25) = \sigma\sqrt{0.5}$$

$$\sigma = \frac{\ln(1.25)}{\sqrt{0.5}} = \frac{0.22314...}{0.70710...} = 0.31557... \cong 31.56\%$$

This means that the values of u = 25% and d = -20% correspond to an annual standard deviation of 31.56%.

In our example, if we wanted to create the a binomial lattice with six, one-month steps, the values that we would use would be:

$$\begin{aligned}
1 + U \equiv u &= e^{(0.3156)\sqrt{1/12}} \\
&= 1.09537... \\
1 + D \equiv d &= 1/u = 0.9129... \\
&\text{or} \\
U &= 9.537...\% \\
D &= -8.70...\%
\end{aligned}$$

Black-Scholes Option Pricing Model

$$C = S \cdot N(d_1) - Xe^{-rT} \cdot N(d_2)$$

$$\left.\begin{aligned}
d_1 &= \frac{\ln\left(\frac{S}{Xe^{-rT}}\right) + \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}} \qquad d_2 = \frac{\ln\left(\frac{S}{Xe^{-rT}}\right) - \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}
\end{aligned}\right\} \text{Brealey and Myers Notation}$$

$$= \frac{\ln\left(\frac{S}{Xe^{-rT}}\right)}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2} \qquad = \frac{\ln\left(\frac{S}{Xe^{-rT}}\right)}{\sigma\sqrt{T}} - \frac{\sigma\sqrt{T}}{2}$$

$$\left.\begin{aligned}
= \frac{\ln\left(\frac{S}{X}\right) + r_f T + \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}} \qquad = \frac{\ln\left(\frac{S}{X}\right) + r_f T - \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}
\end{aligned}\right\} \text{Standard Textbook Notation}$$

$$= d_1 - \sigma\sqrt{T}$$

Applying the Black-Scholes to the example:

$$d_1 = \frac{\ln\left(\frac{40}{40e^{-0.02}}\right)}{0.3156\sqrt{0.5}} + \frac{0.3156\sqrt{0.5}}{2} = 0.2012$$

$$d_2 = 0.2012 - 0.3156\sqrt{0.5} = -0.02194$$

Using the NORMSDIST Excel function:

$N(d_1) = 0.5797$, $N(d_2) = 0.4912$

$C = 40(0.5797) – 40e^{-0.02}(0.4912) = 23.189 – 19.261 = \$3.928$