Many Social Choice Rules

1 Introduction

So far, I have mentioned several of the most commonly–used social choice rules : pairwise majority rule, plurality, plurality with a single run–off, the Borda count. Each of these rules has some weaknesses and/or quirks : pairwise majority rule does not always work (i.e. there may not be an alternative which is a Condorcet winner), plurality neglects alternatives which are popular, but not ranked first by many voters, the Borda count creates incentives for strategic voting.

Soon (in the third section of the course), I will discuss a general feature of all social choice rules : each of them has at least one weakness. Arrow's Impossibility Theorem (and related results which followed it) shows that we will never be able to find a foolproof social choice rule.

Here, I list some of the different social choice rules which have been proposed by various mathematicians, political scientists and cranks — most of which are actually used by some organizations.

2 Approval Voting

Each voter can vote for as many, or as few alternatives as she wishes. Unlike the Borda count, there is no ranking of alternatives, and no point system.

Instead, each voter is asked : of the m alternatives, which are acceptable to you? The voter then lists the alternatives which are acceptable to her. If she wants, she can put just one of the alternatives (presumably her favourite) on the list. In fact, if she wants, she can submit an empty list, indicating she disapproves of all the alternatives. But she could list 2 or 3 or 7 of the alternatives (presumably the 2 or 3 or 7 which are above some threshold level of acceptability). Or all m alternatives, if she wants.

Then we add up the number of votes which each alternative received, and we choose the alternative which has the most votes. That is, the alternative which the most voters find acceptable is the one chosen.

3 Baldwin's Social Choice Rule

We start with all m alternatives, and we compute the Borda count.

But we don't stop there. After this initial calculation, we then eliminate one alternative, the alternative with the **lowest** Borda count.[1] Then we re–do the Borda count, for the remaining $m-1$ alternatives. Again we find the alternative with the lowest Borda count among these $m-1$, and we eliminate this alternative. Then we re–do the Borda count again, for the $m-2$ alternatives still remaining on the ballot.

We keep doing this — $m-1$ times — until there is only one alternative left. This "last alternative standing" is the alternative chosen, under the Baldwin rule.

This rule is quite similar to Nanson's rule below. In fact, Nanson devised his procedure first, and Baldwin proposed his as a modification of Nanson's rule.

4 Black's Social Choice Rule

This rule was proposed by Duncan Black, who is (primarily) responsible for the *Median Voter Theorem*.

He proposed choosing the Condorcet winner, if there was a Condorcet winner.

If there is no Condorcet winner, Black's rule will pick the alternative with the highest Borda count.

5 Bucklin's Voting Rule

Condorcet actually thought of this rule first, but it was discovered independently a century later by James W. Bucklin of Grand Junction, Colorado.

Each voter must provide an ordered list of alternatives, from most–preferred down to least–preferred.

In the first stage, the only information used is the most–preferred alternative of each voter. So we start out with standard plurality voting, counting the number of first–place votes which each alternative gets.

If some alternative gets a majority of the votes, then we are done : that alternative is chosen.

But if no single alternative gets a majority of the votes, then we go into the second stage. We add in all the second–place votes. So the vote total for alternative x in stage 2 is : the number of voters who rank x first **plus** the number of voters who rank x second.

Again, we check whther any single alternative gets a majority of the votes (that is, whether any single alternative is ranked either first or second by a majority of the voters). If the answer is yes, then the alternative with the most votes is chosen. [Note : because each voter gets two votes in this second stage, there might be more than one alternative which gets a majority of votes. We'll

[1] If two or more alternatives are tied for last, we choose one of them, at random, to be eliminated, and leave the rest in.

choose the one with the most votes, from among those alternatives which get a majority.]

If no alternative gets a majority of votes in stage 2, then we go to stage 3, in which we add everyone's third–place votes to their first– and second–place votes.

And we continue in this fashion, until we get to a stage in which at least one alternative gets a majority of the votes. Then the process ends, and the alternative with the most (1st–plus–2nd–plus–3rd–plus ...) votes is chosen.

6 Carey's Voting Rule

This procedure involves a sequence of plurality rule votes.

So we start, in stage 1, with plain plurality rule : each voter votes for her most–preferred alternative.

Then we eliminate all candidates who received **lower–than–average** vote totals. So if there were 300 voters, and 4 alternatives, in the first stage we would eliminate all alternatives with fewer than 75 votes.

Then we run the plurality vote again. Except now, some of the alternatives have been eliminated from the ballot, so the people who voted initially for those alternatives will have to switch their votes, to the alternative still on the ballot which they like best.

After this second stage, we again eliminate all candidates with fewer–than–average vote totals. (So, with 300 voters, if only one of the 4 alternatives had been eliminated in the first ballot, then in the second stage we will eliminate all alternatives which get fewer than $300/3 = 100$ votes.)

We keep eliminating alternatives in this manner, until one alternative gets a majority of the votes. That alternative is the one chosen.

7 Coombs's Voting Rule

This rule was proposed by Clyde Coombs in 1964, and seems to have been borrowed (without attribution) by many (extremely tedious) "reality" shows.

Under Coombs's procedure, each voter ranks all the alternatives, from best to worst. Then the alternative which has the highest number of last–place votes is eliminated. The procedure is then repeated, with the alternative ranked last among the remaining $m - 1$ alternatives in this second stage being eliminated.

The process continues, with the alternative ranked last by the most voters being eliminated in each stage, until one alternative winds up getting a majority of the **first** place votes. That alternative is then chosen.

8 Copeland's Voting Rule

This rule is based on pairwise majority rule.

We calculate an alternative's score by looking at its record in pairwise contests against every other alternative.

For example, suppose that there are 11 alternatives. To calculate the score for alternative x , we look at how it does against each of the other 10 alternatives, in a pairwise election. So if a majority of voters prefer alternative x to alternative y , we say that x wins the contest against alternative y . Alternative x gets 1 point for each win, and -1 points for each loss (with 0 points for a tie). So if alternative x won pairwise elections against 6 of the 10 other alternatives, and lost to the other 4, then its score would be $6 - 4 = 2$.

We rank the alternatives by the scores computed in this way, and choose the alternative with the highest score.

9 Dodgson's Voting Rule

This rule was devised by the British mathematician Charles L. Dodgson, who also wrote *Alice in Wonderland*.

It's an attempt to find the "closest thing" to a Condorcet winner, in cases where there is no Condorcet winner.

So (as in Copeland's rule above), we look at all the pairwise comparisons. We then calculate how close each alternative is to being a Condorcet winner. For example, if alternative w defeats 8 of the other 10 alternatives in pairwise votes, but loses to y by 3 votes and loses to z by 2 votes, then w is 5 votes away from being a Condorcet winner : if 3 voters had changed their ranking of w versus y , and 2 voters had changed their ranking of w versus z , then w would have been a Condorcet winner.

So the alternative chosen is the alternative which is closest to being a Condorcet winner, the alternative for which the fewest of these vote changes would be needed.

To calculate this number of changes , find each pairwise election which the alternative loses, and then add up the margin of defeat over all those losing elections. Dodgson's rule chooses the alternative with the **smallest** number of vote changes needed.

If there is a Condorcet winner, then it will require 0 changes of votes, and so will be chosen under Dodgson's rule.

10 Hare's Voting Rule [a.k.a. "Single Transferrable Vote"]

This rule was devised by a nineteenth–century English barrister, Thomas Hare. It is similar in style to Coombs's rule above (although it predates Coombs's rule by more than a century).

In fact, Hare's rule is plurality with a lot of runoffs, exactly the rule used in many political conventions[2].

[2]Such as the 2012 NDP convention, mentioned in the lecture on "plurality".

We start with plurality voting, each voter voting for a single alternative. Then we eliminate the alternative with the fewest first–place votes, and run another round of plurality voting with only $m - 1$ alternatives on the ballot. Then we eliminate the alternative with the fewest (first–place) votes in that election, and run another round with the remaining $m - 2$ alternatives. The process continues until an alternative gets a majority of the votes, and that alternative is then chosen.

11 Nanson's Voting Rule

This rule is similar to (and precedes) Baldwin's rule described above. Like Baldwin's rule, Nanson's rule involves a sequence of Borda counts.

So we start by computing the Borda count with all m alternatives on the ballot. Then we eliminate all the alternatives which got less than the average score on that ballot. (So if there were 4 alternatives, and 5 voters, the average score under the Borda count would be 12.5, and we would eliminate all alternatives with a Borda count lower than 12.5.)

Then we compute the Borda count again, for the reduced ballot with the remaining alternatives on it which have not yet been eliminated. Again, we find all the alternatives with below–average scores on this ballot, and eliminate them.

We keep repeating this process until only one alternative remains, and that alternative is our choice.

12 Raynaud's Voting Rule

Like Copeland's and Dodgson's rules, Raynaud's rule is based on the pairwise votes between alternatives.

First, we look at all the pairwise votes. So if there were 10 alternatives, there would be 45 different pairs of alternatives. We look at the margin of victory and defeat in each of these pairwise elections. Then we eliminate the alternative which lost by the biggest margin.

We throw that alternative out, and begin again, with only $m-1$ alternatives. We look at all the pairs (36 of them, if $m = 10$), find the biggest margin, and eliminate the loser (that is, the alternative which had the biggest margin of defeat in a pairwise contest). That leaves us with $m - 2$ alternatives.

And we continue this process until we are left with one alternative standing (that is, the winner of the pairwise election between the last two alternatives left standing). And that's our choice.

13 Schulze's Voting Rule

This rule is somewhat complicated. But it is used to choose executives for several organizations[3]

This rule is based on pairwise comparisons. But what distinguishes it from some of the other pairwise–comparison–based rules (such as Copeland's or Dodgson's) is that it allows for **indirect** comparisons.

So suppose that a majority of voters prefer x to y , and that a majority of voters prefer y to z . Then x can defeat z by a sequence of two votes.

If we have a sequence of this nature, let me described it as "x beats z indirectly". That is, x defeats z indirectly if I can find a sequence of pairwise votes, in which x defeats a_1, a_1 defeats a_2, a_2 defeats a_3, and so on, up to a_k defeats z . In the previous paragraph, $k = 1$ and $a_1 = y$.

If I can find such a path, so that x defeats z indirectly, then the margin of victory is defined as the **smallest** of the margins of victory of the k pairwise elections, in the path from z to a_k to a_{k-1} to a_1 to x .

That is, if x defeats y by a vote of 54 to 45, and y defeats z by a vote of 53 to 46, then the margin of victory for this path is 7 : the smaller of the margins in the two votes on the path.

Now the problem with this indirect beating is that we can have x defeating z indirectly, but also z defeating x indirectly. That's what the Condorcet paradox showed.

So the way we're going to rank alternatives when this cycling occurs is to look at the margin of victory. We define $p(x , z)$ as the largest margin of victory from all the indirect victories x has over z .

If z cannot defeat x indirectly — that is if there is no path of pairwise votes which gets us from x to z — then define $p(z , x)$ as equalling 0.

Finally, we use this margin of victory to rank our alternatives. We'll rank x above z if and only if $p(x , z) > p(z , x)$.

And our choice will be any alternative x which is ranked above all other alternatives.

There are two very nice features about this measure $p(x , z)$. First, it's transitive. If $p(x , y) > p(y , x)$ and $p(y , z) > p(z , y)$ then it must be true that $p(x , z) > p(z , x)$. Second, there must always be a "Schulze winner" : there must be some alternative x such that $p(x , a) \geq p(a, x)$ for every other alternative a.

So this second property ensures that we can use this measure of margin of victory to choose a "best" alternative. And the first property ensures that we can also use this measure of margin of victory to generate a social ordering.

14 Simpson's Voting Rule

Simpson's rule predates Schulze's. It's simpler, but also seems to have more potential weaknesses.

[3] Wikimedia, and Debian Gnu/Linux software, for example.

Simpson's rule is based on pairwise votes. For alternative x , we take all the pairwise contests against the other $m-1$ alternatives, and we find the biggest defeat x suffers. So if x defeats 8 of the other 10 candidates in a pairwise vote, but loses to w by 3 votes, and loses to z by 2 votes, then the biggest defeat margin for x is 3.

We then rank the alternatives in reverse order of this largest margin of defeat. We choose the alternative for which the largest margin of defeat is smallest.

15 Small's Voting Rule

Small's rule is a refinement of Copeland's. Copeland's rule looked simply at the number of alternatives that x defeated. It ranks the candidates in order of the number of pairwise contests they win.

The problem is, that very often we'll have ties.

Small's rule helps break some of those ties. It starts with Copeland's rule. We look at all pairwise votes, and give a score of $+1$ to the winner, -1 to the loser and 0 if it's a tie. Then we add up each alternative's score over all the pairwise votes it has, and rank the alternatives by these total scores.

Then, if more than one alternative is tied for best overall, we do another round. We eliminate all the alternatives that are not tied for best, anbd re–calculate the score, using only the pairwise votes among the alternatives remaining.

If this second stage results in a tie, then we eliminate all alternatives which are not tied for the top, and recalculate again. And we keep doing this until there is only one alternative left (that will be our social choice), or until there are several alternatives left, and they're all tied with each other (those will be our choices).

16 Tideman's Voting Rule

Tideman's rule is another rule based on pairwise votes. We look at all of the pairwise votes (45 of them if there were 10 alternatives), and we try to use the margin of victory as a basis for generating a ranking.

So we look at all the pairwise votes, and we find the one with the biggest margin. We "lock" that vote. That is, if the biggest margin is in the vote of x versus y , and it's won by x , we'll lock x in above y . Then we look for the next–largest margin. Say it's the victory of y over z . So next we lock that vote in, ranking y above z .

Now suppose the third–largest victory is z over x . We'd like to rank z above x , but that would generate a problem with transitivity, since we already have locked in x above y , and y above z . So we ignore this third result. That is, whenever a result contradicts the rankings that we have already locked, then we ignore that result. We move on to the 4th–biggest margin, say w over z , and if that doesn't contradict any rankings we've already generated, we'll lock that in, ranking w above z .

So we keep going down the list of margins of victory, using them to rank pairs of alternatives — unless this would generate a cycle when combined with the rankings we already have. And we keep doing this until we have generated a complete top–to–bottom ranking. That's our social ordering, and our choice will be the top alternative in this ordering.