1 Calculus - Optimization - Applications

The task of finding points at which a function takes on a local maximum or minimum is called *optimization,* a word derived from applications in which one often desires to find the the optimal value of a certain system which is being modeled mathematically. In the business setting we will look for an optimal value i.e. maximum value of a function that measures say profit or revenue.

In the previous chapter we showed that if a function has a local maximum or a minimum at a point $x = c$, then provided the derivative $f'(c)$ exists, we can conclude that the derivative is zero at c. This result gives us a starting point in a search for maximums and minimums. The points where the derivative is zero are likely candidates for local maximums or minimums. We need however further analysis, for at each such a point the function may take on a maximum value, a minimum value or neither. Fortunately, there is an obvious method.

1.1 Increasing/decreasing functions

The method for distinguishing local maximums and minimums is based on the fact that if a function attains a local maximum value at a point c, then on the left of the point c, the values $f(x)$ get gradually larger as x approaches c , whereas, on the right of c, the values of $f(x)$ get gradually smaller with increasing values of x greater than c. A similar observation applies to points where f attains a local minimum value.

To make this more precise lets say that a function is *increasing* on an interval I, provided the functional values $f(x)$ increase with increasing x and that it is *decreasing* if the functional values decrease with increasing x. To be absolutiely precise we have the following definition. See figure 1.

Definition 1 Increasing / Decreasing Fucntions
A function f defined on an interval I is increasing on I if for every $x_1 \in I$ and for every $x_2 \in I$, if $x_1 < x_2$, then $f(x_1) < f(x_2)$. The function f is decreasing on I, if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

With this idea at hand we can now say that a function f has a local maximum at c, if in some neighborhood about c, the function is increasing on the left of c and decreasing on the right of c. Similarly, f has a local minimum at c if the function is decreasing on the right and increasing on the left. See figure 2.

This is all well and good, but how can we determine if a function is either increasing or decreasing. Geometrically the answer is clear. If the tangents to the graph of the function all have positive slope, then geometrically it is evident that the function itself must be

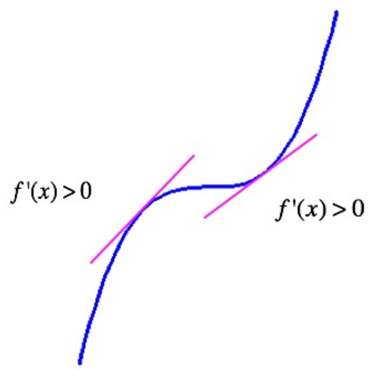

Figure 1: if derivatives do not change sign, there is no extremum

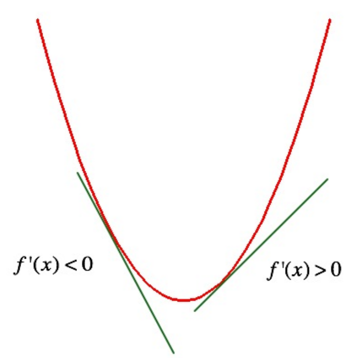

Figure 2: for minimum derivatives change from positive to negative

increasing. Again draw some pictures to convince yourself. Likewise if the tangents have negative slope, the function must be decreasing . So for a function f defined on an interval I, this then means that if $f'(x) > 0$ for all $x \in I$, then the tangents all have positive slope and the function is increasing on I. Similarly if $f'(x) < 0$ for all $x \in I$, then the function is decreasing on I. To prove this result analytically requires a bit of work that is best postponed to a full course in calculus, but it is in fact this geometric intuition that guides the proof and is what one relies on in practice.

Lemma 2 *Let $f : I \to \mathbb{R}$ be a function defined on an interval I. If $f'(x) > 0$ for each x in I, then f is increasing on I. If $f'(x) < 0$ for each $x \in I$, then f is decreasing on I.*

In figure 1 which is the graph of the function $f(x) = x^3$, it is easy to verify that $f'(x) = 3x^2$ is greater than zero for $x < 0$ and for $x > 0$. The function is increasing on both of these intervals, and as we can see there is neither a local maximum or a minimum at $x = 0$. On the other hand, in figure 2, which is the graph of $f(x) = x^2$, one can show that $f'(x) = 2x$ is less than zero for $x < 0$ and $f'(x) > 0$ for $x > 0$, and as one can see there is a local minimum at $x = 0$.

1.2 The First derivative test

Combining our ideas, we can then say that a function f has an extremum at a point c, if the sign of the derivative changes from positive to negative at c, in the case of a maximum, and in the case of a minimum, if it changes from negative to positive. More precisely we have the *First Derivative Test.* Later we will also look at the so called *Second Derivative Test* which abstracts information by looking at the derivative of the derivative - in other words, the second derivative.

Result 3 First Derivative Test
Given a function f which is defined on an open interval (a, b), and let c be such that $a < c < b$. Then

1. *if $f'(x) > 0$ for $a < x < c$ and $f'(x) < 0$ for $c < x < b$, then at the point c the function $f$has a local maximum.*
2. *if $f'(x) < 0$ for $a < x < c$ and $f'(x) > 0$ for $c < x < b$, then at the point c the function $f$has a local minimum.*
3. *if $f'(x)$ does not change sign at c, then there is neither a maximum or a minimum at c*

It is important to realize that the first derivative test provides a method for determining local maximums or minimums at a point c even when the derivative does not exist at c. Points at which the derivative is zero or does not exist are referred to as *critical points.*

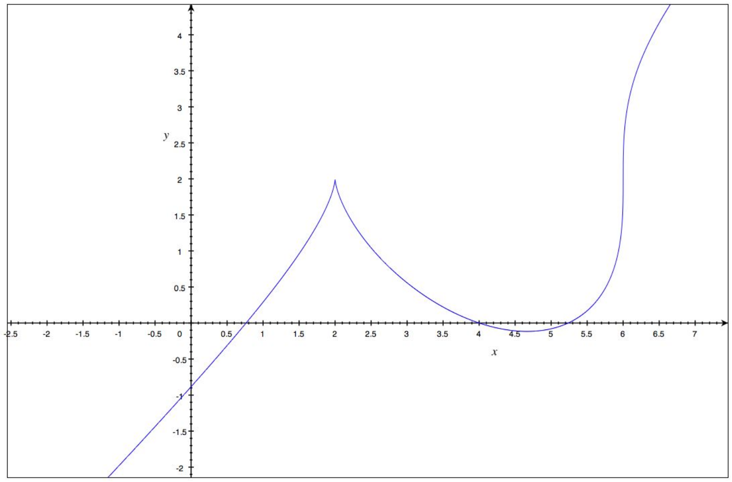

Figure 3: $f(x) = -(x-2)^{\frac{2}{3}}(6-x)^{\frac{1}{3}} + 2$ has a cusp and a local maximum at $x = 2$

And in the hunt for local maximums and minimums one always has to check not only the points at which the derivative is zero but also where the derivative does not exist. The most obvious example of a function which has an extremum at a point where the derivative does not exist is that of the absolute value function $f(x) = |x|$, whose graph looks like a **V**. For $x < 0, f'(x) = -1$ and the functional values are decreasing and for $x > 0, f'(x) = 1$ and the functional values are increasing. However, the derivative of f does not exist at $x = 0$. This behavior holds whenever a function has a *cusp* ; namely a point $(c, f(c))$ where the left derivative taken by considering the left limit of the difference quotient $\lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h}$ is not equal to the right limit of the difference quotient $\lim_{h\to 0+} \frac{f(c+h)-f(c)}{h}$. Or, in other words, where distinct left and right tangent lines can be constructed. See figure 3.

1.3 Sign analysis and applying the first derivative test

Lets see now how to apply the first derivative test to the function

$$f(x) = -(x-2)^{\frac{2}{3}}(6-x)^{\frac{1}{3}} + 2$$

whose graph we see in figure 3. In particular we would like to see that the information from the first derivative test agrees with what we see in the graph. Computing the first derivative we have:

$$\begin{aligned}
\frac{d}{dx}\Big(-(x-2)^{\frac{2}{3}}(6-x)^{\frac{1}{3}}+2\Big) &= -\tfrac{2}{3}\frac{(6-x)^{\frac{1}{3}}}{(x-2)^{\frac{1}{3}}}+\tfrac{1}{3}\frac{(x-2)^{\frac{2}{3}}}{(6-x)^{\frac{2}{3}}} \\
&= \tfrac{1}{3}\Big(\frac{-2(6-x)^{\frac{1}{3}}}{(x-2)^{\frac{1}{3}}}+\frac{(x-2)^{\frac{2}{3}}}{(6-x)^{\frac{2}{3}}}\Big) \\
&= \tfrac{1}{3}\Big(\frac{-2(6-x)+(x-2)}{(x-2)^{\frac{1}{3}}(6-x)^{\frac{2}{3}}}\Big) \\
&= \tfrac{1}{3}\Big(\frac{3x-14)}{(x-2)^{\frac{1}{3}}(6-x)^{\frac{2}{3}}}\Big)
\end{aligned}$$

Examining the result we see that for $x = 2$ or $x = 6$, we end up with a zero in the denominator, which leaves us with the derivative not defined at these points. And in the numerator, setting it equal to zero and solving for x, gives $x = 4\frac{2}{3}$ as a possible local minimum or maximum. To gain any more information, we need to apply the 1st derivative test, and to do that we need to be able to determine its sign, positive or negative in the vicinity of the points $x = 2, x = 6$, and $x = 4\frac{2}{3}$.

There is a method for doing this which I refer to as *sign analysis*. It goes like this - for a function described as a product of various terms, the sign of the function for any specific value of x is determined by the sign of its various factors. So to determine the sign of $f'(x)$ for any arbitrary value of x, separate the factors and determine how the sign of each varies as a function of x. From this one can determine the sign of the product as x varies.

For the function at hand, I note that there are three factors, $3x-14$, $\frac{1}{(x-2)^{\frac{1}{3}}}$, and $\frac{1}{(6-x)^{\frac{2}{3}}}$ But the sign of the latter two is determined by the sign of their reciprocals, $(6-x)^{\frac{2}{3}}$ and $(x-2)^{\frac{1}{3}}$, which are easier to work with. In the table shown in figure 4, I show the method. The top line of the table is meant to correspond to the real line, and I've labeled the critical points $2, 4\frac{2}{3}$, and 6. In the first row, the minus signs indicate that $3x - 14$ is negative for all x less than $4\frac{2}{3}$. The plus signs in the first row indicate that $3x - 14$ is positive for all x greater than than $4\frac{2}{3}$. The other cells in the first three rows are filled in similarly. Note that since $(6-x)^{\frac{2}{3}}$ is a square of a cube root, it will always be postive. The last row tells the sign of $f'(x)$ as x varies, and we obtain it simply by remembering that the product of a positive and a negative is negative, the product of two negatives is negative, and so forth.

Reading from the table, we now see that the sign of the first derivative changes from positive to negative at $x = 2$ and then from negative to positive at $x = 4\frac{2}{3}$. Thus according to the first derivative test there is a maximum of the function at $x = 2$ and a minimum at $x = 4\frac{2}{3}$. Furthermore, this corresponds to what appears to be the case in figure 3.

$-\infty$ $\qquad 2 \qquad 4\frac{2}{3} \qquad 6 \qquad \infty$

$3x-14$	-	-	+	+
$(x-2)^{\frac{1}{3}}$	-	+	+	+
$(6-x)^{\frac{2}{3}}$	+	+	+	+
$f'(x)$	+	-	+	+

Figure 4: sign analysis for $\dfrac{3x-14)}{(x-2)^{\frac{1}{3}}(6-x)^{\frac{2}{3}}}$

1.4 Concavity and the second derivative test

The first derivative test can be a bit messy in practice, for checking to see if $f'(x)$ is greater or less than zero, takes some work. The second derivative test is usually quick and easy but depends on f having second derivatives. And, it turns out that there are plenty of examples of functions for which the first derivative exists but not the second. The beauty behind the second derivative test is that by examining the sign or the second derivative, one can determine basic properties of the graph. There are two basic notions - that are best understood by pictures. A portion of a graph of a function is *concave down over an interval I,* if for each $x \in I$ this portion of the graph lies below the line tangent to the graph at the point $(x, f(x))$, see figure 5 The graph over I is *concave up,* if it lies above each such tangent line, see figure 2.

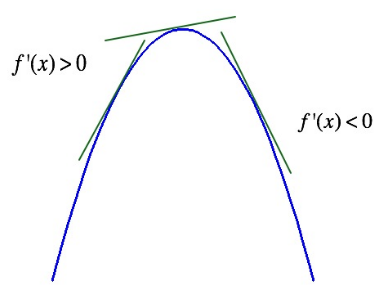

Figure 5: concave down - slopes of tangents are decreasing

The sign of the second derivative lets us determine if the graph is concave up or concave down over an interval I. To see why this is true, we need to look at the behavior of the first derivative. Look at an example where the graph is clearly concave down, for instance the graph in figure 5 and look at the slopes of the tangent lines to points $(x, f(x))$ as x increases. What do you see? The slopes are getting progressively smaller. Starting at the left of a maximum point the slopes are positive, gradually getting smaller with increasing x. At the maximum point the slope is zero, and then they continue, as negative numbers, getting yet smaller, as we see in figure 5. So it appears evident that if the first derivative values are decreasing, the graph must be concave down. Okay fine and good, but how can we determine if the first derivative is decreasing? Think back. Considering the first derivative $f\,'$ as just another function, we have already shown that a function is decreasing on an interval provided its derivative takes on negative values at each point of the interval. Therefore, if the derivative of $f\,'$, namely $f\,''$ is less than zero on on I, that is, $f\,''(x) < 0$ for $x \in I$, then $f\,'$ is decreasing, and the graph is concave down over I. The same analysis works to show that if $f\,''(x) > 0$ on I, then f is concave up over I.

Result 4 Test for concavity
Given a function f, if $f\,''(x) < 0$ for x in an interval I, then the graph of f is concave down over I and if $f\,''(x) > 0$ for x in I, then the graph of f is concave up over I

Lets now apply the notion of concavity to the problem of finding local maximums and minimums. If we have determined that $f\,'(c) = 0$, then we know that the tangent to the graph at $(c, f(c))$ is horizontal, even if in briefest sense as is the case of the cubic $f(x) = x^3$, when $x = 0$. If however we also know that the graph concave down in a region about $(c, f(c))$, then there must be a maximum at $x = c$, whereas if the graph is concave up f must have a minimum at $x = c$. Combining these observations with our ability to determine concavity with the sign of the second derivative, we have an effective way of determining maximums and minimums. However in order to insure that the function is concave down or concave up in some reasonable region about $(c, f(c))$, we will add the extra condition that the second derivative be continuous in some small interval containing c. This means that if $f\,''(c) > 0$, the second derivative must remain greater than zero in some small region and therefore graph will remain concave down in the same region.

Result 5 Second derivative test
Let f be a function and suppose that $f\,'(c) = 0$ and that the second derivative $f\,''$ is continuous at c. Then:

1. *if $f\,''(c) > 0$, then f has a local minimum at $x = c$.*
2. *if $f\,''(c) < 0$, then f has a local maximum at $x = c$.*

Example 6 *Using the concavity test and the second derivative test, for the function $f(x) = 3x^5 - 4x^3 + 1$, find the local maximums and minimums and the intervals over which the graph is concave down and the intervals where the graph is concave up.*

First we need to find the critical points where the first derivative $f\,'$ is zero. Then we could substitute these values into the second derivative f'' to determine if they are local maximums or minimums. However, we are also asked to find the regions of concavity, so instead, we will find:

- the intervals where $f''(x)$ is negative and thus where the graph is concave down
- the intervals where $f''(x)$ is positive and thus where the graph is concave up.

Computing the first derivative, we have

$$f'(x) = 15x^4 - 12x^2.$$

Setting it equal to zero and solving for x, we get

$$15x^4 - 12x^2 = 3x^2(5x^2 - 4) = 0,$$

which gives $x = 0$ or $x = \pm\sqrt{4/5} = \pm\sqrt{0.8}$ as possible sites for a maximum or minimum Computing the second derivative gives

$$f''(x) = 60x^3 - 24x.$$

We need to find where f'' changes sign, and to do this we need to find the points where it is zero. Setting $f''(x)$ equal to zero and solving for x, observe that gives

$$60x^3 - 24x = 12x(5x^2 - 2) = 60x(x^2 - \frac{2}{5}).$$

Setting the latter expression equal to zero and solving for x then gives solutions $x = 0$ or $x = \pm\sqrt{\frac{2}{5}} = \pm\sqrt{0.4}$.

Next, to find the intervals where f'' is positive or negative we can make use of the sign analysis technique on the intervals determined by the solutions. The factors to consider that will determine the sign of $f''(x)$ are: $60x, x - \frac{2}{5}$, and $x + \frac{2}{5}$. And for simplicity the constant multiple of 60 can be omitted from the first factor as it will not influence the sign of the first term. We have the sign analysis table in figure 6. This tells us that the graph of f is concave down on the intervals $(-\infty,\ -\sqrt{0.4}$ and $(0,\ \sqrt{0.4}\,)$ and that it is concave up on the intervals $(-\sqrt{0.4},\ 0)$ and $(\sqrt{0.4},\ \infty)$.

Finally, checking the zeros of the first derivative, since $-\sqrt{\frac{4}{5}} = -\sqrt{0.8}$ lies in the interval $(-\infty,\ -\sqrt{0.4}\,)$, we then know that $f''\Big(-\sqrt{\frac{4}{5}}\,\Big) < 0$ so that by the second derivative test f has a local maximum at $x = -\sqrt{\frac{4}{5}}$. Similarly, since $\sqrt{\frac{4}{5}} = \sqrt{0.8}$ lies in the interval $(\sqrt{0.4},\ \infty)$, it follows that $f''\Big(\sqrt{\frac{4}{5}}\,\Big) > 0$ so f has a local minimum at $x = \sqrt{\frac{4}{5}}$.

All fine and good, but what about the point $x = 0$. The sign analysis table in figure 6 shows that $x = 0$ is a transition point between the graph being concave down and concave

$-\infty$ $\quad -\sqrt{0.4} \quad 0 \quad \sqrt{0.4} \quad \infty$

$x+\sqrt{0.4}$	−	+	+	+
x	−	−	+	+
$x-\sqrt{0.4}$	−	−	−	+
$f''(x)$	−	+	−	+

Figure 6: sign analysis for $f''(x) = 60x^3 - 24x$

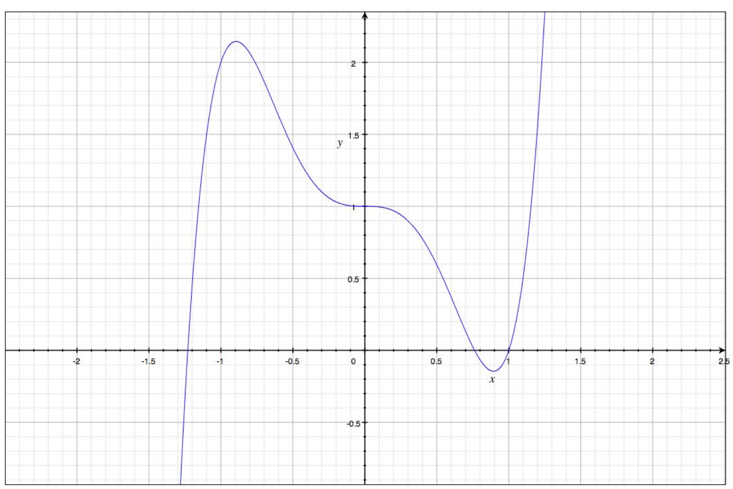

Figure 7: graph of $f(x) = 3x^5 - 4x^3 + 1$

up. As such it cannot be a local maximum or minimum. Such transition points are called *inflection points.*

What we have discovered is corroborated by the the graph of the function which is shown in figure 7.

1.4.1 Inflection points

In the example of the previous section the notion of an inflection point was introduced as a transition point in a graph, whereby on side of the point the graph is concave down and on the other it is concave up. More precisely there is the following definition.

Definition 7 *Given a function f, a point c is an inflection point of f if there is an interval (a,b) containing c such that: (1) the graph concave down over the interval (a,c) and concave up over (c,b) or (2) the graph is concave up on (a,c) and concave down on (c,b).*

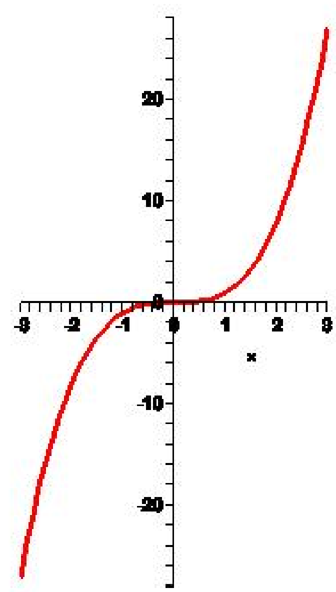

Figure 8: $f(x) = x^3$ changes concavity at 0

If we look at the graph of $f(x) = x^3$, figure 8, we see that at $x = 0$ is an inflection point since the graph clearly changes there from concave down to concave up. Such a point is called an *inflection point*.

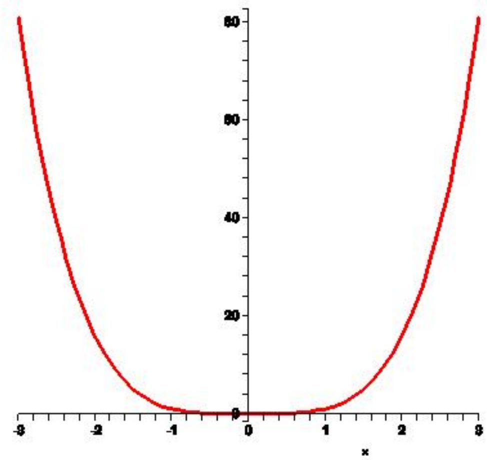

Figure 9: $f(x) = x^4$ has $f''(0) = 0$ but does not change concavity at 0

If f is a function, using the concavity test there is a simple criterion for determining inflection points. Suppose the second derivative f'' is continuous in some interval containing a point c and that the sign of changes at c. Then by the concavity test, the point c is an inflection point, but in addition, because of the continuity of f'' and the fact that f'' has one sign on one side of c and the opposite sign on the other side of c, it follows that f'' must be zero at $x = c$ - i.e. $f''(c) = 0$. To summarize we have the following result.

Result 8 *If a function f has an inflection point at $x = c$ and if the second derivative f'' is continuous in some interval containing c, then $f''(c) = 0$*

Hunting for inflection points is then much like hunting for maximums and minimums. Find where $f''(x) = 0$, then if at one of these points, say $x = c$, the sign of f'' changes from negative to positive or positive to negative, you have an inflection point at $x = c$. To see that it is not sufficient to simply look at the points where $f'(x) = 0$, consider the example of $f(x) = x^4$. Calculating the second derivative, we see that $f'(x) = 4x^3$, so that $f''(x) = 12x^2$. Clearly $f''(0) = 0$, but the sign of the second derivative does not change at $x = 0$, and the graph is evidently concave up; see figure 9.

1.4.2 Examples

Following are a number of examples showing how our techniques can be used to find optimal solutions to a variety of problems

1. Suppose 1500 square centimeters are available for constructing a cardboard box con-

sisting of a square base and an open top. What is the largest possible volume of the box?

Solution: Let h stand for the height of the box and let x stand for the length of an edge of the square base. The total amount of material can be calculated in terms of x and h and equated to 1500. Then one can solve for one of x or h in terms of the other, and with this one can construct the volume equation, say $V(x)$, and search for a maximum.

The four sides have together an area of $4xh$ and the bottom has an area of x^2. Thus the total area is $4xh+x^2 = 1500$, and $h = \dfrac{1500-x^2}{4x} = \dfrac{375}{x} - \dfrac{x}{4}$. Calculating volume, we have $V = x^2 h$, so as a function of x, we have

$$\begin{aligned} V(x) = & \quad x^2\Big(\frac{375}{x} - \frac{x}{4}\Big) \\ = & \quad 375x - \frac{1}{4}x^3. \end{aligned}$$

We then have $V'(x) = 375 - \frac{3}{4}x^2$. Setting $V'(x) = 0$ and solving for x gives $375 - \frac{3}{4}x^2 = 0$ and $x^2 = 500$. Thus $x = \pm 10\sqrt{5}$, and because of the nature of the problem, we only consider the positive solution $x = 10\sqrt{5}$.

Next we check the second derivative to see if $x = 10\sqrt{5}$ represents a minimum or a maximum for the function V. We calculate $V''(x) = -\frac{3}{2}x$, and since $V''(10\sqrt{5}) < 0$, we conclude that V has a maximum at $x = 10\sqrt{5}$.

2. A farmer in Ontario has 100 hectares which he wishes to fence as a rectangular field subdivided into two smaller fields with a fence parallel to one of the sides. What should the dimensions be that will minimize the cost of the fencing?

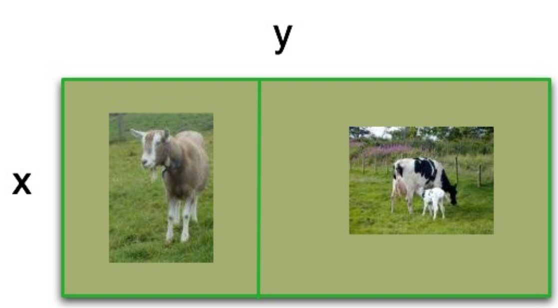

Figure 10: rectangular field divided in two

Solution: Let the field have dimensions x meters and y measured in meters, and divide it by a fence parallel to the side of length x. The amount of fencing to be used is then $3x + 2y$ meters. Since 1 hectare $= 10^4$ square meters, 100 hectares then equals 10^6 square meters. Thus the area of the field is $xy = 10^6$ so that $y = \frac{10^6}{x}$ and the total amount of fencing then represented by the function,

$$f(x) = 3x + \frac{2 \times 10^6}{x}.$$

Taking the derivative gives $f'(x) = 3 - \dfrac{2 \times 10^6}{x^2}$. Setting $f'(x) = 0$ gives

$$\begin{aligned}
3 &= \frac{2 \times 10^6}{x^2} \\
3x^2 &= 2 \times 10^6 \\
x^2 &= \tfrac{2}{3} \times 10^6 \\
x &= \pm\sqrt{\tfrac{2}{3}} \times 10^3 \quad \approx 816.5
\end{aligned}$$

Again we consider only the positive solution $x = \sqrt{\frac{2}{3}} \times 10^3$, and we check to see if f takes on a maximum value here by checking the second derivative. We find that, $f''(x) = \dfrac{4 \times 10^6}{x^3}$, and we see that $f''(x) > 0$ for all positive values of x, including $x = \sqrt{\frac{2}{3}} \times 10^3$. Thus by the second derivative test f has a minimum at this point. We then have:

$$\begin{aligned}
y = &\quad \frac{10^6}{\sqrt{\frac{2}{3}} \times 10^3} \\
= &\quad \frac{10^3}{\sqrt{\frac{2}{3}}} \\
= &\quad \frac{10^3}{\sqrt{\frac{2}{3}}} \cdot \frac{\sqrt{\frac{2}{3}}}{\sqrt{\frac{2}{3}}} \\
= &\quad \frac{3}{2}\sqrt{\frac{2}{3}} \times 10^3 \;\approx 1,224.6.
\end{aligned}$$