Chapter 3
Decision theory

3.1 INTRODUCTION

Decision theory deals with methods for determining the optimal course of action when a number of alternatives are available and their consequences cannot be forecast with certainty.

It is difficult to imagine a situation which does *not* involve such decision problems, but we shall restrict ourselves primarily to problems occurring in business, with consequences that can be described in dollars of profit or revenue, cost or loss. For these problems, it may be reasonable to consider as the best alternative that which results in the highest profit or revenue, or lowest cost or loss, on the average, in the long run. This criterion of optimality is not without shortcomings, but it should serve as a useful guide to action in repetitive situations where the consequences are not critical. (Another criterion of optimality, the maximization of expected "utility," provides a more personal and subjective guide to action for a consistent decision-maker.)

The simplest decision problems can be resolved by listing the possible monetary consequences and the associated probabilities for each alternative, calculating the expected monetary values of all alternatives, and selecting the alternative with the highest expected monetary value. The determination of the optimal alternative becomes a little more complicated when the alternatives involve sequences of decisions.

In another class of problems, it is possible to acquire—often at a certain cost—additional information about an uncertain variable. This additional information is rarely entirely accurate. Its value—hence, also the maximum amount one would be willing to pay to acquire it—should depend on the difference between the best one expects to do with the help of this information and the best one expects to do without it.

These are, then, the types of problems which we shall now begin to examine in more detail.

3.2 DECISION PROBLEMS

Very simply, the *decision problem* is how to select the best of the available alternatives. The elements of the problem are the possible *alternatives (actions, acts)*, the possible *events (states, outcomes of a random process)*, the *probabilities* of these events, the *consequences* associated with each possible alternative-event combination, and the *criterion (decision rule)* according to which the best alternative is selected.

Example 3.1 A grocery receives its weekly supply of eggs every Thursday morning. This shipment must last until the following Thursday when a new shipment is received. Any eggs left unsold by Thursday are destroyed. Eggs sell for \$10 per hundred and cost \$8 per hundred. The weekly demand for eggs at this grocery varies from week to week. From past experience, the following probability distribution is assigned to weekly demand:

Demand (hundreds of eggs):	10	11	12	13	14
Probability:	0.1	0.2	0.4	0.2	0.1

This pattern of demand remains stable throughout the year—the demand for eggs is not seasonal, and the trend is flat. The problem is: How many eggs should be ordered for delivery every Thursday?

The possible alternatives, the possible events, the probabilities of these events, and the associated consequences are shown in a *payoff table*, Table 3.1.

Table 3.1
Payoff table
(Entries show profit in hundreds of dollars)

Events (hundreds of eggs demanded)	Probability	Alternatives (hundreds of eggs ordered) 10	11	12	13	14
10	0.1	20	12	4	−4	−12
11	0.2	20	22	14	6	−2
12	0.4	20	22	24	16	8
13	0.2	20	22	24	26	18
14	0.1	20	22	24	26	28
Expected profit:		20	21*	20	15	8

To illustrate the construction of this table, suppose that 12 (hundred) eggs are ordered. The purchase cost is 12×8 or 96 (hundreds of dollars). If 10 (hundred) eggs are demanded, 10 are sold, and the revenue is 10×10 or

100 (\$00); the profit associated with this alternative-event pair is $100 - 96$ or 4 (\$00). Similarly, if demand is 12, the profit is $(12 \times 10) - (12 \times 8)$ or \$24. But if demand is greater than 12, only 12 can be sold, and the profit remains \$24. We assume that no additional penalty—such as loss of goodwill—is incurred if demand is not met.

Consider now the first alternative. If 10 are ordered, the profit will be \$20 no matter what demand happens to be. In contrast, the last alternative offers a chance of realizing a profit of \$28 but also a chance of a \$12 loss, as well as chances of in-between profit values. In fact, the profit of each alternative may be represented by a random variable, which is a function of the random variable representing demand. For example, if we let Y be the random variable representing the profit of the third alternative, the probability distribution of Y is:

y	$p(y)$
4	0.1
14	0.2
24	0.7
	1.0

Thus, if 12 eggs are ordered every Thursday morning, in the long run, a profit of \$4 will be realized in 1 week out of 10, a profit of \$14 in 2, and a profit of \$24 in 7 out of 10 weeks. The *expected profit*, that is, the long-run average profit, is simply the mean of Y:

$$(4)(0.1) + (14)(0.2) + (24)(0.7) = 20.$$

The expected profit of all alternatives is shown in the last row of the payoff table. In this case, it would be quite reasonable to use expected profit as the criterion for selecting the best alternative. The alternative with the highest average profit in the long run is also the one with the highest long-run total profit. The second alternative, with an expected profit of \$21, is the best alternative under this criterion.

Our definition of a decision problem applies, of course, in many situations. In a sense, what one cooks for dinner, whom one marries, which clothes one buys, ... all these are decision problems. Not all decision problems will concern us here. For the most part, we shall restrict ourselves to business problems in which the consequences are measured in dollars of profit, revenue, or cost, and in which a reasonable criterion of optimality is to select the alternative with the highest *expected monetary value*, that is, the highest expected profit or revenue, or the lowest expected cost. These problems will tend to be of the repetitive kind, so that the probabilities may be interpreted as the long-run relative frequencies, and the *expected payoff* as the average payoff in the long run.

A similar criterion of optimality, however, can be applied to a wider class of decision problems. As will be explained in the next section, if the decision-maker is able to act in accordance with certain reasonable rules of conduct in uncertain situations, then the best act in *any* decision problem is the act with the highest *expected utility.* The expected utility of an act is calculated like the expected profit or cost, after replacing each consequence by its "utility"—roughly, by the personal value of that consequence for the decision-maker.

3.3 SHORTCOMINGS OF EXPECTED MONETARY VALUE, UTILITY

In this section, we examine non-recurring decision problems in which the consequences cannot be measured in monetary terms. In theory, at least, these problems may be resolved by establishing the "utilities" of the consequences, subjectively estimating the probabilities of the possible events, and selecting the act with the highest expected utility. Two examples will illustrate the nature of the problem and the method of resolution.

Example 3.2 You are considering buying a ticket for a certain lottery. The ticket costs \$100 and the lottery will be conducted only once. This is a rather crude lottery: a coin will be tossed; if it turns up heads, you will receive \$250; if it turns up tails, you will get nothing. Should you buy this ticket or not?

		Alternatives (Acts)	
Events	Probability	Buy	Do not buy
Heads	0.5	+150	0
Tails	0.5	−100	0
Expected profit:		25	0

The consequences shown in the payoff table represent profit, i.e., the difference between revenue and cost. If you buy the ticket and the coin turns up heads, the profit is \$250 − \$100, or \$150; if you buy and tails shows up, your profit if \$0 − \$100, or −\$100. If you do not buy the ticket, the profit is \$0, regardless of what the outcome of the toss might have been. Assuming that the coin is fair, the probabilities of heads and tails are 0.5 each. The expected profit of the act Buy is \$25, while that of Do Not Buy is \$0. The first act has the highest expected profit, and, according to this criterion, is the best act.

Yet many people would not agree that buying the lottery ticket is the best act. They may point out that the situation is not repetitive, since the lottery is conducted once only; therefore, the expected profit cannot be interpreted as the long-run average profit. They may also point out that, if the ticket is bought, the probability is 1/2 of a profit of \$150, but there is an equal probability of a loss of \$100. In their minds, the prospect of losing \$100 may loom larger than that of gaining \$150.

Assuming that the assessment of the probabilities is not questioned, the fact that different people reach different decisions in this situation may be due to their attaching different values to the monetary consequences.

Example 3.3 You are preparing a three-egg omelette. Having already broken two good eggs into the pan, you are suddenly assailed by doubts about the quality of the third—as yet unbroken—egg. Two things may happen: either the egg is good or it is rotten. The three possible acts and the associated consequences are shown in Table 3.2.

Table 3.2
The omelette problem

Events	Probability	Break third egg into pan	Break third egg into a saucer and inspect	Throw away third egg
Third egg is good	0.9(?)	3-egg omelette	3-egg omelette, one saucer to wash	2-egg omelette, one good egg destroyed
Third egg is rotten	0.1	No omelette, two good eggs destroyed	2-egg omelette, one saucer to wash	2-egg omelette

This is a somewhat frivolous example, but it illustrates two points. First, it *is* a decision problem, albeit one in which the consequences cannot be given numerical values—monetary or otherwise. Second, people *do* resolve this and similar problems by weighing—consciously or unconsciously—the values of the consequences and the probabilities of their occurrence. Different people may assess differently these values, depending on their preferences and experience.

We shall first state the procedure for determining the utilities of the consequences, illustrating with data from Example 3.2.

1. Determine the most preferred and the least preferred consequence. To these consequences assign utilities of 1 and 0 respectively.

In Example 3.2, clearly the most preferred consequence is the profit of \$150, and the least preferred is the loss of \$100. Thus, the utility of \$150 becomes 1 and the utility of $-$\$100 becomes 0.

2. For each of the remaining consequences, specify the probability for which you would be indifferent between that consequence, on the one hand, and a lottery, on the other hand, having two outcomes: the most preferred consequence, with the stated probability, and the least preferred consequence, with 1 minus the stated probability. The specified probability becomes the utility of the consequence.

The only remaining consequence in Example 3.2 is \$0, that is, the consequence of doing nothing. For which probability (p) would *you* be indifferent between "doing nothing" and a lottery which with probability p will give you \$150, and with probability $1 - p$ will result in your paying \$100? When p is close to 0, that is, when the loss of \$100 is virtually certain, you would prefer to do nothing. On the other hand, when p is close to 1, that is, when the gain of \$150 is virtually assured, you would prefer the lottery. Somewhere between 0 and 1 there is a value of p for which you are indifferent between doing nothing and the lottery. Different people will come up with different answers. Let us suppose that your answer is 0.8. In other words, you are indifferent between \$0 and the gamble: \$150 with probability 0.8, $-$\$100 with probability 0.2. Then, the utility of \$0 for you is 0.8.

3. Replace all consequences by their utilities. Calculate the expected utility of each act, and select the act with highest expected utility.

For Example 3.2, the utility table is as follows:

Events	Proba-bility	Acts: Buy	Acts: Do not buy
Heads	0.5	1	0.8
Tails	0.5	0	0.8
Expected utility:		0.5	0.8

The optimal act is the act with the highest expected utility, that is, the act Do Not Buy.

Example 3.3 (Continued) The most preferred consequence is the three-egg omelette; the least preferred is no omelette. For each remaining consequence you must specify the probability, p, for which you are indifferent between that consequence for sure and a lottery which with probability p will give you a three-egg omelette, and with probability $1 - p$ will result in no omelette. Again, the preferences of different people will not be the same. Let us suppose that you specify the following probabilities:

Consequences	Probability of indifference, p, and utility
3-egg omelette, one saucer to wash	0.9
2-egg omelette	0.6
2-egg omelette, one saucer to wash	0.5
2-egg omelette, one good egg destroyed	0.4

The utility table is as follows:

Events	Proba-bility	Break third egg into pan	Break third egg into a saucer and inspect	Throw away third egg
Third egg good	0.9	1	0.9	0.4
Third egg rotten	0.1	0	0.5	0.6
Expected utility:		0.9	0.86	0.42

The first act has the highest expected utility and is optimal.

Any decision problem can be resolved by this approach. What is required of the decision-maker is an expression of his or her preferences in choices involving consequences and the "reference" lotteries described above.

But why *should* anyone follow this prescription for determining the optimal act? In an environment in which any decision-maker is free to select that alternative which he or she finds most appealing, the justification of a normative rule must be conditional on some fundamental principles. If the decision-maker agrees that certain rules of conduct are reasonable, then he or she ought to agree that the implications of these rules are also reasonable. This, in fact, is the major contribution of modern utility theory. For it can be shown that a decision-maker who follows consistently certain reasonable rules of conduct acts *as if* he or she assigns utilities to consequences in the manner described in this section and selects the act with the highest expected utility. Space does not allow us to discuss fully these rules of conduct and their implications. The reader will find a more thorough treatment in specialized references.

3.4 SOME SHORTCUTS

It is not always necessary to list all possible events, alternatives, and consequences in a payoff table. The act with the highest expected profit, or lowest expected cost, can sometimes be found directly by exploiting the nature of the relationship among consequences, acts, and events, as the following will demonstrate.

Linear payoff functions. Let us suppose that the events may be represented by the values of a random variable, X, with probability distribution $p(x)$.

Payoff table

Events x	Proba-bility	$\cdots$	Act i	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
x	$p(x)$	$\cdots$	$c(x) = a + bx$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
Expected payoff:		$\cdots$	$E[c(X)] = a + bE(X)$	$\cdots$

Let us also suppose that the payoff (profit, revenue, cost or loss) of a certain act given that $X = x$, $c(x)$, is a linear function of X,

$$c(x) = a + bx,$$

where a and b are some constants. The expected payoff of this act is

$$\begin{aligned} E[c(X)] &= \sum_x c(x)p(x) \\ &= \sum_x (a + bx)p(x) \\ &= a\sum_x p(x) + b\sum_x xp(x) \\ &= a + bE(X). \end{aligned}$$

In words, the expected payoff of the act is the payoff of the expected value of the random variable. (This result applies to continuous, as well as discrete, random variables.)

Example 3.4 The manufacturer of a certain product is considering the purchase of one of three different packaging systems. The product sells for \$10, and the production cost (excluding packaging cost) is \$5 per unit. The cost data for the three packaging systems are:

System No.:	Purchase cost	Variable cost per unit of product	Scrap value
1	\$100	\$1.50	\$10
2	200	1.00	20
3	400	0.50	40

All three systems last one year only and will then be sold at the listed salvage value. The demand for the product over the year can be regarded as a random variable with the following probability distribution:

Demand, x	Probability, $p(x)$
100	0.3
200	0.6
400	0.1
	1.0

Which system should be bought?

First note that the expected demand is

$$E(X) = (100)(0.3) + (200)(0.6) + (400)(0.1) = 190.$$

There are three alternatives. If system 1 is bought, the profit—the difference between revenue and cost—is the following linear function of demand, x:

$$c_1(x) = (10x + 10) - (5x + 1.5x + 100) = -90 + 3.5x.$$

Similarly, for system 2 the profit is

$$c_2(x) = (10x + 20) - (5x + 1.0x + 200) = -180 + 4x,$$

and for system 3,

$$c_3(x) = (10x + 40) - (5x + 0.5x + 400) = -360 + 4.5x.$$

Therefore, the expected profit of each alternative is:

$$E[c_1(X)] = -90 + 3.5E(X) = -90 + (3.5)(190) = 575,$$

$$E[c_2(X)] = -180 + 4E(X) = -180 + (4)(190) = 580^*,$$

and

$$E[c_3(X)] = -360 + 4.5E(X) = -360 + (4.5)(190) = 495.$$

The best act is to buy system 2. The expected profit of this act is \$580.

3.5 A SINGLE-STAGE INVENTORY PROBLEM

A merchant must decide how many units to purchase of a certain perishable product. He buys the product at $\$c$ per unit at the beginning of the period (year, month, week, etc.) and sells it during the period for $\$p$ per unit. Any stock remaining unsold at the end of the period has no value and is discarded. The decision problem is to select q, the optimum number of units to purchase. We suppose that the demand for the product during the period can be regarded as a random variable, X, with probability distribution $p(x)$.

The principal feature of this problem is that the product is *perishable* and cannot be carried over to the next period. This is the type of problem facing fashion designers, manufacturers or buyers of specialized equipment, food retailers, and other handlers of perishable products. The possible alternatives are the possible values of q (0, 1, 2, ...), and the possible events are the possible values of demand, X (0, 1, 2, ...). We assume that the demand and stock levels are discrete, and that the initial inventory is zero.

We shall show below that *the optimum number of units to purchase is the largest value of q for which*

$$Pr(X < q) \quad < \quad \frac{p-c}{p},$$

and *the expected profit of purchasing q units is*

$$p\sum_{x=0}^{q}(x-q)p(x) + q(p-c).$$

The proof of this statement is not easy. The reader interested in an application of the optimal rule may skip the following on first reading.

¶ Suppose q (q being any specified number—not necessarily the optimal one) units are purchased. Since sales cannot exceed the quantity available,

$$\text{Sales} = \begin{cases} x & \text{if} \quad x \leq q, \\ q & \text{if} \quad x > q. \end{cases}$$

Therefore, the profit for the period is

$$\text{Profit} = \begin{cases} px - cq & \text{if} \quad x \leq q, \\ pq - cq & \text{if} \quad x > q. \end{cases}$$

The expected profit is

$$
\begin{aligned}
f(q) &= \sum_{x=0}^{q}(px - cq)p(x) + \sum_{x=q+1}^{\infty}(pq - cq)p(x) \\
&= p\sum_{x=0}^{q} xp(x) - cq\sum_{x=0}^{q} p(x) + pq\sum_{x=q+1}^{\infty} p(x) - cq\sum_{x=q+1}^{\infty} p(x) \\
&= p\sum_{x=0}^{q} xp(x) + pq[1 - \sum_{x=0}^{q} p(x)] - cq \\
&= p\sum_{x=0}^{q}(x-q)p(x) + q(p-c).
\end{aligned}
$$

If, instead of q, $q-1$ units are purchased, the expected profit is

$$
\begin{aligned}
f(q-1) &= p\sum_{x=0}^{q-1}(x-q+1)p(x) + (q-1)(p-c) \\
&= p\sum_{x=0}^{q-1}(x-q)p(x) + p\sum_{x=0}^{q-1}p(x) + q(p-c) - (p-c) \\
&= p\sum_{x=0}^{q}(x-q)p(x) - p(q-q)p(q) + p\sum_{x=0}^{q-1}p(x) + q(p-c) - (p-c) \\
&= [p\sum_{x=0}^{q}(x-q)p(x) + q(p-c)] + [p\sum_{x=0}^{q-1}p(x) - (p-c)] \\
&= f(q) + [pPr(X<q) - (p-c)],
\end{aligned}
$$

and

$$f(q) - f(q-1) = (p-c) - pPr(X<q).$$

Thus, the incremental profit is positive (and q is better than $q-1$) as long as

$$p - c \;\; > \;\; pPr(X<q),$$

or

$$Pr(X<q) \;\; < \;\; \frac{p-c}{p}.$$

This is the optimality condition stated earlier. Note that if q^* is the largest value of q for which $Pr(X<q) \;<\; (p-c)/p$, then $Pr(X<q) > (p-c)/p$ for all $q > q^*$. In other words, once the incremental profit $(p-c) - pPr(X<q)$ becomes negative, it stays negative. The proof is now complete. ¶

Example 3.1 (Continued) This is an example of a single-stage inventory problem. The optimal solution was found earlier following the standard method. We shall now show how the same solution may be obtained, with fewer calculations, using the results of this section. In this example, X represents demand, $p = 10$, and $c = 8$. Therefore, $(p - c)/p = (10-8)/10 = 0.2$, and

q	$Pr(X < q)$
10	0.0
11*	0.1
12	0.3
13	0.7
14	0.9
15	1.0

The largest value of q for which $Pr(X < q)$ is less than 0.2 is $q^* = 11$. Thus, the optimum number to stock is 11 units. The expected profit of this alternative is:

$$\begin{aligned} f(q^*) &= p \sum_{x=0}^{q^*} (x - q^*)p(x) + q^*(p - c) \\ &= (10)[(10 - 11)(0.1) + (11 - 11)(0.2)] + (11)(10 - 8) \\ &= 21. \end{aligned}$$

These results are, of course, identical to those obtained earlier, but fewer calculations were required to arrive at the optimal solution.

3.6 DECISIONS IN STAGES, DECISION TREES

In some cases, the choice of the optimal act is not made in one stage, and the decision problem involves a sequence (not necessarily in time) of acts, events, acts, events, etc. There may be a number of basic alternatives, each leading to one of a number of situations depending on the outcome of a certain random process. At each such situation, a number of other alternatives may be available which also lead to a new set of situations depending on another set of events ... and so on, with acts followed by events, followed by acts, events, etc. The sequence of acts and events may be depicted in the form of a *decision tree*. The decision problem is to find the most preferred branch of that tree.

Example 3.5 A product is manufactured by a certain automatic machine in batches of 3 items. The selling price is \$10, and the production cost is \$5 per item. Before each batch is produced, the machine may be adjusted by a skilled mechanic at a cost of \$6 per batch. If the machine is so adjusted, there will be no defectives in the batch. If it is not adjusted, some items may be defective; on the basis of past experience, the probability distribution of the number of defectives is estimated as follows:

Number of defectives in batches of 3 items	Probability
0	0.4
1	0.3
2	0.2
3	0.1
	1.0

Defective items may be sold as scrap for \$5 each, or may be reprocessed at the following cost schedule:

Number of reprocessed items	Total reprocessing cost
1	\$ 6
2	10
3	12

Reprocessed items can be sold as good items. Should the machine be adjusted before each batch is produced? Figure 3.1 shows the decision tree for this problem.

The two basic alternatives are Adjust and Do Not Adjust. If the machine is adjusted, there will be no defective items; the revenue is \$30, the total cost is \$21, and the profit is \$9. If the machine is not adjusted, there may be 0, 1, 2, or 3 defectives in the batch, with probabilities 0.4, 0.3, 0.2, and 0.1 respectively.

Suppose that the machine is not adjusted, and the batch contains 2 defectives. Two alternatives are available: Reprocess (RP), or Sell (S) the defectives as scrap. (For simplicity, we assume that *both* defectives must either be reprocessed or sold as scrap; we do not consider the possibility that one must be reprocessed and the other sold as scrap.) If the defectives are reprocessed, there will be 3 good items to sell and the revenue will be \$30; the total cost is the sum of the production and the reprocessing cost, or \$25; the profit of the act Reprocess is \$5. If the two items are sold as

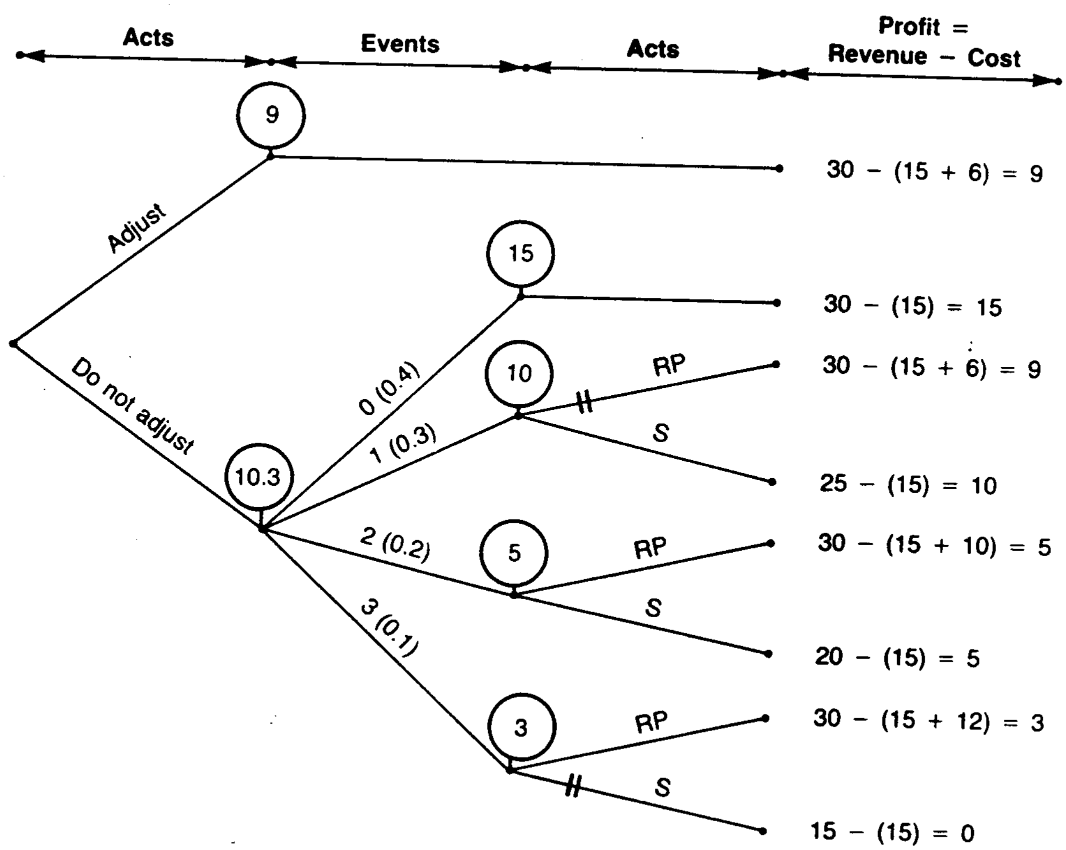

Figure 3.1
Decision tree, Example 3.5

scrap, the revenue is \$20, the cost is \$15, and the profit is \$5. Therefore, if two defectives occur, one is indifferent between these two alternatives.

The evaluation of the other four situations is shown in Figure 3.1. Thus, if there are no defectives, the profit is \$15. If the batch contains one defective, the best act is Scrap and the profit of this act is \$10. If three defectives occur, the best act is Reprocess and the profit of this act is \$3. In Figure 3.1, non-optimal acts are excluded, and the profit of the optimal act (circled) is carried back to the preceding node.

Therefore, if the machine is not adjusted, the profit will be \$15 with probability 0.4, or \$10 with probability 0.3, or \$5 with probability 0.2, or \$3 with probability 0.1. The expected profit of the act Do Not Adjust is

$$(15)(0.4) + (10)(0.3) + (5)(0.2) + (3)(0.1),$$

or \$10.30. Since the profit of the act Adjust is \$9, the optimal basic act is not to adjust the machine. The optimal *strategy* is: Do not adjust the machine; scrap, if two or fewer defectives occur; reprocess, if more than two defectives occur.

This example illustrates an approach which may be followed in general for solving problems of this type. Lay out the sequence of acts and events in the form of a decision tree. Calculate the consequences at the end of every branch of the tree. At each point *(decision node)* where choice is provided, eliminate non-optimal alternatives. Calculate the expected payoff at each point *(event node)* where events branch out. Work backwards in this fashion until the basic alternatives are evaluated.

3.7 MULTISTAGE INVENTORY PROBLEMS

In Example 3.1 we examined a very simple inventory problem. It was made simple by assuming that any inventory left unsold at the end of any period could not be carried over to the next period. In reality, of course, the inventory problem is a multiperiod one. If demand is less than the quantity available for sale, the surplus can be used to meet demand in the next period; whatever is left at the end of the second period can be used during the third period; and so on. A simple example will illustrate the new concepts.

Example 3.6 A retailer regularly stores a certain product so as to meet his customers' requirements. On the basis of past experience, he estimates the following probability distribution of weekly demand, that is, the number of units of the product required by customers in any one week:

Demand, d	Probability, $p(d)$
0	0.1
1	0.3
2	0.4
3	0.2
	1.0

The retailer would like to adopt an ordering policy that will maximize his profit over a number of weeks. The product sells for \$6 per unit. The purchase cost consists of a fixed component (\$2) incurred every time an order is placed, and a variable component (\$3) for every unit of the product purchased. We assume that orders are filled instantly. At the beginning of each week, the retailer checks his stock and decides whether to order, and, if so, how many units to order. The initial stock, plus the new purchases (if any), constitute the quantity available for sale during the week. If demand is lower than the quantity available for sale, sales equal the quantity demanded, and the inventory at the end of the week equals the difference between the quantity available and sales. We assume that the cost of carrying this inventory to the next week is \$1.50 per unit of ending inventory; this represents the cost of storage plus the cost of tying down a certain amount of capital in inventory. If, on the other hand, demand exceeds the quantity available, the excess demand is lost. We assume that the resulting loss of customers' goodwill amounts to \$2.50 for every unit demanded but not available.

Let us further suppose that the retailer follows a so-called (s, S) *policy*. Under this policy, whenever the inventory on hand is less than or equal to s units, $[S-(\text{Inventory on hand})]$ units are ordered; when inventory on hand exceeds s, no order is placed.

Obviously, the number of such policies is very large. We assume the retailer would like to find that (s, S) policy which maximizes the total expected profit over a planning period of, say, three weeks, given that his initial inventory is 0. Any units of the product left unsold at the end of the third week can be disposed of at half-price, i.e., at \$3 per unit.

Let us begin by calculating the expected profit of a *given* (s, S) policy, namely, the policy $s = 1$, $S = 2$. This means that if inventory at the beginning of any week is less than or equal to 1 (i.e., if it is 0 or 1), the quantity ordered is equal to $[2-(\text{initial inventory})]$; if the initial inventory is greater than 1, no orders are placed.

To calculate the expected profit of this policy over three weeks, we shall work backwards in time. We shall first assume that we stand at the beginning of the *third* week and shall determine the expected profit for that week as a function of the initial inventory.

Table 3.3
Expected profit for week 3, given inventory at beginning of week 3 is 1
(1 unit is ordered; 2 units available for sale)

Demand	Probability	Sales	Lost sales	Ending inv.	Revenue	Cost: Hold.	Cost: Short.	Cost: Purch.	Cost: Total	Week's profit	Fut. exp. profit	Total profit
0	0.1	0	0	2	0.00	3.00	0.00	5.00	8.00	−8.00	6.00	−2.00
1	0.3	1	0	1	6.00	1.50	0.00	5.00	6.50	−0.50	3.00	2.50
2	0.4	2	0	0	12.00	0.00	0.00	5.00	5.00	7.00	0.00	7.00
3	0.2	2	1	0	12.00	0.00	2.50	5.00	7.50	4.50	0.00	4.50
	1.0									Expected total profit:		4.25

Table 3.3 shows the calculations in the case where the inventory at the beginning of the third week is equal to 1. For example, suppose that demand

is 1. Sales equal 1, the revenue is \$6, the total cost is \$6.50, and the profit is \$−0.50. The disposal of 1 unit of ending inventory at half-price brings in an additional \$3 in revenue, and the total profit is \$2.50. The probability that this profit will occur is the probability (0.3) that demand will equal 1. The total profit corresponding to all other possible demand levels is calculated in the same way. The expected total profit for week 3, if the initial inventory is 1, is

$$(-2.00)(0.1) + (2.50)(0.3) + (7.00)(0.4) + (4.50)(0.2) = \$4.25.$$

Table 3.4
Expected profit for week 3, given inventory at beginning of week 3 is 3
(no orders; 3 units available for sale)

Demand	Probability	Sales	Lost sales	Ending inv.	Revenue	Cost: Hold.	Cost: Short.	Cost: Purch.	Cost: Total	Week's profit	Fut. exp. profit	Total profit
0	0.1	0	0	3	0.00	4.50	0	0	4.50	− 4.50	9.00	4.50
1	0.3	1	0	2	6.00	3.00	0	0	3.00	3.00	6.00	9.00
2	0.4	2	0	1	12.00	1.50	0	0	1.50	10.50	3.00	13.50
3	0.2	3	0	0	18.00	0.00	0	0	0.00	18.00	0.00	18.00
	1.0									Expected total profit:		12.15

If the inventory at the beginning of the third week is 3 units, the expected total profit is \$12.15, as shown in Table 3.4. The reader can verify that the expected total profit is \$1.25 if the initial inventory is 0, and \$9.25 if it is 2. These calculations are summarized below:

Inventory at beginning of week 3, x	Expected total profit, $Q_3(x)$
0	1.25
1	4.25
2	9.25
3	12.15

Let us now move back to the beginning of the *second* week, and consider the consequence of following this inventory policy over weeks 2 and 3. Suppose that the inventory at the beginning of week 2 is 0. Two units are ordered to make the quantity available for sale equal to 2. The purchase cost is (2) + (3)(2) or \$8. The expected total cost over weeks 2 and 3 is calculated in Table 3.5.

Table 3.5
Expected profit for weeks 2 and 3, given inventory at beginning of week 2 is 0
(2 units ordered; 2 units available for sale)

Demand	Probability	Sales	Lost sales	Ending inv.	Revenue	Cost: Hold.	Cost: Short.	Cost: Purch.	Cost: Total	Week's profit	Fut. exp. profit	Total profit
0	0.1	0	0	2	0.00	3.00	0.00	8.00	11.00	−11.00	9.25	−1.75
1	0.3	1	0	1	6.00	1.50	0.00	8.00	9.50	− 3.50	4.25	0.75
2	0.4	2	0	0	12.00	0.00	0.00	8.00	8.00	4.00	1.25	5.25
3	0.2	2	1	0	12.00	0.00	2.50	8.00	10.50	1.50	1.25	2.75
	1.0									Expected total profit:		2.70

For example, if demand is 3, sales are 2, the revenue in week 2 is \$12, the total cost for week 2 is \$10.50, the profit for the week is \$1.50, and week 3 begins with an inventory of 0. Our earlier calculations show that the expected profit for week 3 is \$1.25 if the initial inventory is 0. Therefore, the expected total profit over weeks 2 and 3, given that demand in week 2 is 3, is \$2.75. The probability of this occurrence is the probability that demand in week 2 will be 3 units, which is 0.2. The other entries in Table 3.5 are similarly calculated. When the inventory at the beginning of week 2 is 0, therefore, the expected total profit over weeks 2 and 3 is

$$(-1.75)(0.1) + (0.75)(0.3) + (5.25)(0.4) + (2.75)(0.2) = \$2.70.$$

The reader can verify that the expected profit over weeks 2 and 3 for each possible level of initial inventory in week 2 is as summarized below:

Inventory at the beginning of week 2, x	Total expected profit over weeks 2 and 3, $Q_2(x)$
0	2.70
1	5.70
2	10.70
3	14.19

The expected total profit over *three* weeks, given an initial inventory and demand level in week 1, is equal to the sum of the profit for week 1 and the expected profit over weeks 2 and 3 associated with the implied inventory at the end of week 1. As shown in Table 3.6, the expected total profit over three weeks starting with an inventory of 0 units is \$4.15.

These calculations are illustrated in Figure 3.2. Since the inventory policy is given, there is only one possible act at each decision node; namely,

Table 3.6
Expected profit for weeks 1, 2 and 3, given inventory at beginning of week 1 is 0
(2 units ordered; 2 units available for sale)

Demand	Proba-bility	Sales	Lost sales	Ending inv.	Reve-nue	Cost Hold.	Cost Short.	Cost Purch.	Cost Total	Week's profit	Fut. exp. profit	Total profit
0	0.1	0	0	2	0.00	3.00	0.00	8.00	11.00	−11.00	10.70	−0.30
1	0.3	1	0	1	6.00	1.50	0.00	8.00	9.50	− 3.50	5.70	2.20
2	0.4	2	0	0	12.00	0.00	0.00	8.00	8.00	4.00	2.70	6.70
3	0.2	2	1	0	12.00	0.00	2.50	8.00	10.50	1.50	2.70	4.20
	1.0									Expected total profit:		4.15

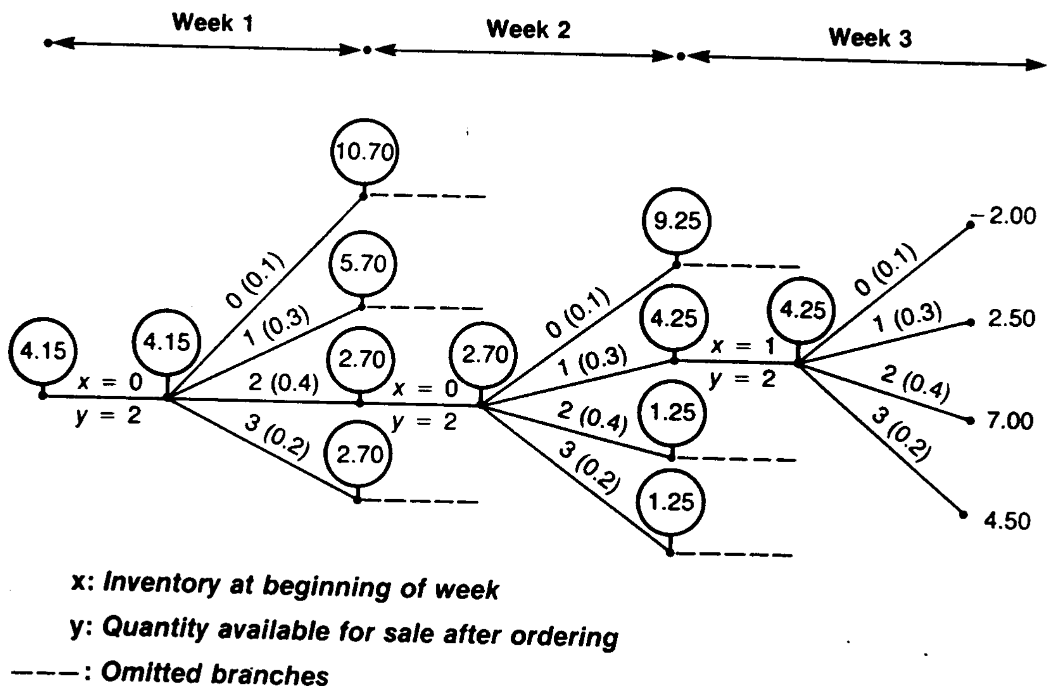

Figure 3.2
Evaluation of inventory policy ($s = 1$, $S = 2$)

to order in accordance with the ($s = 1$, $S = 2$) policy. Other policies could be shown by expanding the number of branches at each decision node.

The method by which we may calculate the expected profit of *any specified* inventory policy over *any* number of periods should be fairly clear. We begin with the last period, and calculate the expected profit of the policy for each possible level of inventory at the beginning of this period. Next, we move back one period, and calculate the expected profit over two periods for each inventory level at the beginning of that period. In the same way, we move backwards to the first period.

3.8 THE VALUE OF ADDITIONAL INFORMATION

We would now like to consider some situations in which it is possible, before an action is taken, to obtain additional information regarding which event will occur. This additional information is not always accurate or free. The problem is to determine if the expected benefits from using the additional information exceed the cost of obtaining it, and, if so, by how much. We shall develop a method of approaching problems of this type with the help of a simplified example.

During the summer months, Vendex operates a concession at River Park, selling ice cream, candy, hot dogs, soft drinks, etc. It employs young people of high school age who are hired as a group, by the day, at $500 per day. On weekdays, business is fairly stable. If the weather is good, the day's revenue is $2,000; if there is rain, the day's revenue drops to $500. At this time of the year, rain tends to occur in two out of every ten days. The cost of the products sold is approximately 30% of the revenue. Rent, maintenance, taxes, the manager's salary, and other fixed costs amount to about $400 per day.

We restrict our analysis to weekdays—a separate analysis must be made for weekends. Each day, Vendex has two alternatives: to operate the concession with all employees, or to remain closed. (In order to keep this example as simple as possible, we do not consider varying the number of employees or the number of hours worked.) The payoff table is as follows:

		Alternatives	
Events	Probability	Open	Close
Rain	0.2	−550	−400
No rain	0.8	+500	−400
Expected profit:		290	−400

For example, if the concession is open and there is no rain, the revenue will be $2,000, the cost of materials 2000×0.3, or $600, the labor cost $500, and the fixed cost $400; therefore, the day's profit is $2000 - (600 + 500 + 400)$ or $500.

If Vendex could know that it was going to rain, it would close. If it could know that it was not going to rain, it would open. Not knowing whether rain will occur or not, the best alternative is to open, because this has the highest expected profit, \$290.

Each day, a forecast is available of next day's weather. Vendex could use that forecast to decide whether to open or close tomorrow. Unfortunately, these forecasts are not always accurate, as the following table shows:

Actual weather	Forecast "Rain"	"No rain"	Total
Rain	0.13	0.07	0.20
No rain	0.02	0.78	0.80
Total	0.15	0.85	1.00

The entries of this table show the joint relative frequencies of actual and forecast weather. For example, in 13% of past summer days, the forecast was "Rain" and rain did occur; in 7% of the days, the forecast was "No rain" but there was rain; and so on. (In order to distinguish actual and forecast weather categories, the latter are written in quotation marks.) Assuming that it is reasonable to use these joint relative frequencies as joint probabilities, the questions are: (a) Should Vendex take advantage of the weather forecasts? (b) If so, how much are the forecasts worth?

To answer both questions, we ought to compare Vendex's expected profit *with the aid* of the forecasts and that *without the aid* of the forecasts. The problem is outlined in Figure 3.3. (The numbers shown in Figure 3.3 are explained below.)

As we saw earlier, without the aid of the forecasts, the best alternative is to open every day and the expected profit of this alternative is \$290.

In order to determine the expected profit of acting with the aid of the weather forecasts, we examine two cases.

(i) Forecast is "Rain." In the past, rain was forecast for 15% of the days. Rain actually occurred in 0.13/0.15 or 87% of these days; in 0.02/0.15 or 13% of these days there was no rain. Thus, the conditional probability of rain given that the forecast is "Rain" is 0.87, and that of no rain is 0.13. The alternatives and the consequences remain the same as in the first payoff table. The only change is in the probabilities of the two events. The expected profits are calculated in the following table:

Events	Probability	Alternatives Open	Close
Rain	0.87	−550	−400
No rain	0.13	+500	−400
Expected profit:		−413.5	−400*

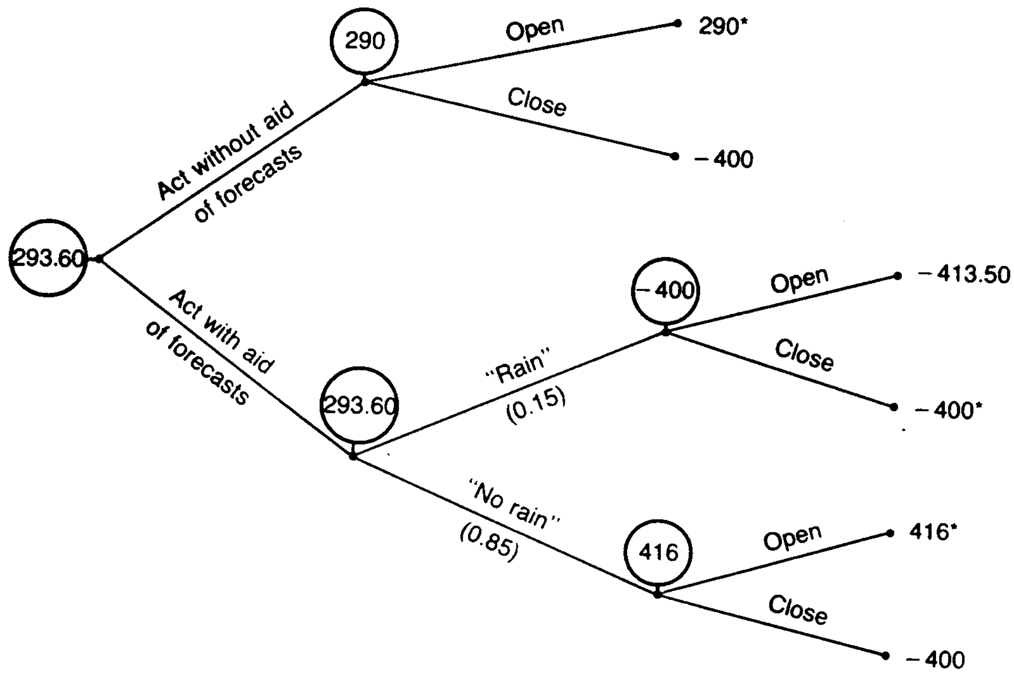

Figure 3.3
Value of additional information, weather forecasting example

Therefore, if the forecast is "Rain," the best alternative is to close and the expected profit of this alternative is $−400.

(ii) Forecast is "No rain." The conditional probability of rain given that the forecast is "No rain" is 0.07/0.85 or 0.08, while that of no rain is 0.78/0.85 or 0.92. In this case, the payoff table is as follows:

		Alternatives	
Events	Probability	Open	Close
Rain	0.08	−550	−400
No rain	0.92	+500	−400
Expected profit:		416*	−400

Therefore, if the forecast is "No rain," the best alternative is to open, and the expected profit of this alternative is $416.

It remains only to calculate the overall expected profit of acting with the aid of the weather forecasts. According to the data, rain was forecast on 15% and no rain on 85% of the days. Thus, with probability 0.15, the forecast is "Rain," the best act is to close, and the expected profit of this act is −400; with probability 0.85, the forecast is "No rain," the best act is to open, and the expected profit of that act is $416. Summarizing:

Forecast	Probability	Best act	Expected profit of best act
"Rain"	0.15	Close	−400
"No rain"	0.85	Open	+416
	Expected profit:		293.60

The overall expected profit of acting with the aid of the forecasts is (0.15)(−400) + (0.85)(416) or $293.60.

The expected profit of acting without the aid of the forecasts is $290. It is, therefore, more profitable to utilize the weather forecasts, but the difference between the expected profits of the two courses of action is only $3.60. The weather forecasts are worth $3.60 per day to Vendex; Vendex should not pay more than $3.60 per day for them.

3.9 A MATHEMATICAL FORMULATION

The approach described in the previous section can be generalized. Suppose that the consequences of an act depend on events which may be represented by the values (or categories) of a random variable (or attribute) X with probability distribution $p(x)$. Suppose further that the consequences are expressed in dollars of cost, so that the best act is that with the lowest expected cost. (The following results can be easily modified for the case where the consequences are expressed in dollars of profit or revenue.) The cost of the "typical" act i when X equals x is written as $c_i(x)$.

			Acts	
Events	Probability	$\cdots$	i	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
x	$p(x)$	$\cdots$	$c_i(x)$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
Expected cost:		$\cdots$	$E[c_i(X)]$	$\cdots$

We wish to examine if another variable, Y, can be utilized to improve decisions. We assume that $c_i(x)$ does not depend on Y. Let $p(x, y)$ be the joint probability that $X = x$ and $Y = y$, $p(x)$ the marginal probability that $X = x$, and $p(y)$ the marginal probability that $Y = y$:

		Y		
X	$\cdots$	y	$\cdots$	Total
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
x	$\cdots$	$p(x,y)$	$\cdots$	$p(x)$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
Total	$\cdots$	$p(y)$	$\cdots$	1.0

We shall compare the expected cost of acting *with the aid* of Y to that of acting *without the aid* of Y.

(a) Without the aid of Y, the expected cost of act i is

$$E[c_i(x)] = \sum_x c_i(x)p(x).$$

Act, say, m is the optimal act if it has the lowest expected cost, i.e., if

$$\sum_x c_m(x)p(x) = \min_i[\sum_x c_i(x)p(x)]. \tag{3.1}$$

The expected cost of acting without the aid of Y is the expected cost of this act.

(b) To evaluate the overall expected cost of acting with the aid of Y, we begin by assuming that $Y = y$. The conditional probability that $X = x$ given $Y = y$ is

$$p(x|y) = \frac{p(x,y)}{p(y)}.$$

The conditional expected cost of act i given that $Y = y$ is

$$\sum_x c_i(x)p(x|y).$$

The optimal act when $Y = y$ depends on y. Let us denote this best act as $k(y)$. In other words,

$$\sum_x c_{k(y)}(x)p(x|y) = \min_i[\sum_x c_i(x)p(x|y)].$$

The overall expected cost of acting with the aid of Y is

$$\sum_y p(y)[\sum_x c_{k(y)}(x)p(x|y)]. \tag{3.2}$$

The expected value of the information provided by Y is the difference between the expected cost of acting without the help of Y and the expected

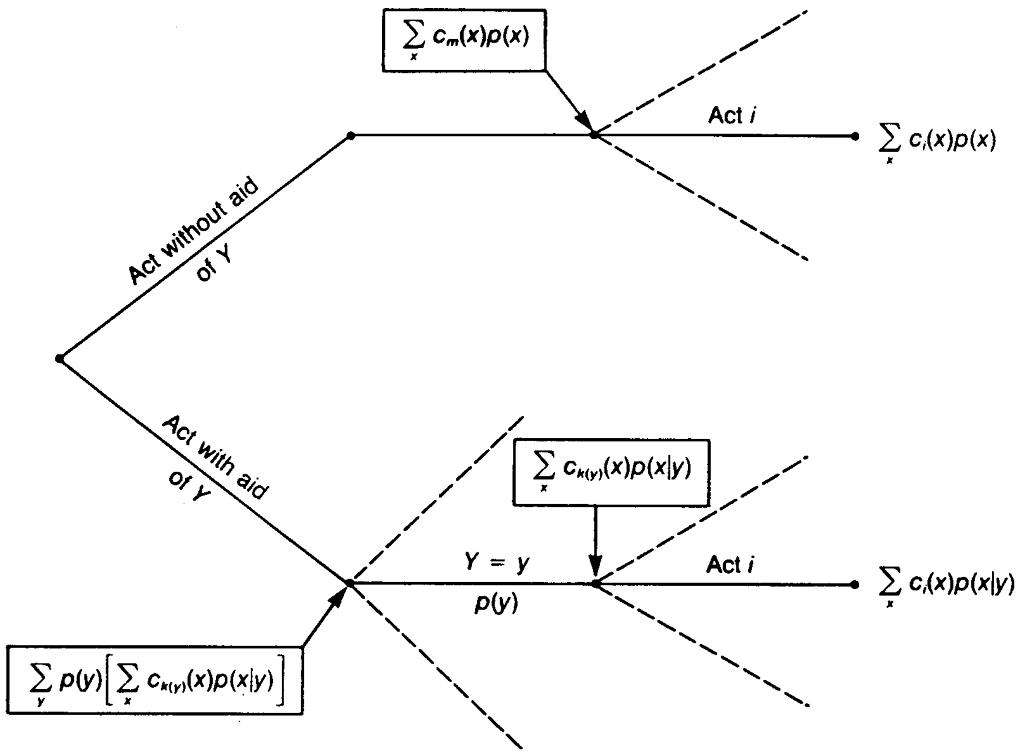

Figure 3.4
The value of additional information

cost of acting with the help of Y, or, Equation (3.1) *minus* Equation (3.2). Figure 3.4 outlines the nature of the problem.

It can be shown that the expected value of the information provided by Y *cannot be negative.* Intuitively, this is quite reasonable; for one of our choices when acting with the aid of Y is to ignore Y. It follows that the best we can do with the help of Y cannot be worse than the best we can do without its help. Therefore, the expected value of the information provided by Y is always zero or positive.

In most cases, the use of Y is not free but must be paid for (for example, when weather forecasts are available by subscription only). It will be profitable to use Y only when the expected value of the information provided by Y exceeds the fee for using Y. The former amount is the maximum fee that should be paid for the use of Y.

3.10 BAYES' FORMULA

In evaluating the decision to act with the aid of Y, we need the conditional probabilities that $X = x$ given $Y = y$,

$$p(x|y) = \frac{p(x,y)}{p(y)}.$$

These conditional probabilities can always be obtained from the joint distribution of X and Y—if this is available. In some cases, however, this joint distribution is not given. Instead, we are given $p(x)$, the marginal probabilities that $X = x$, and *can infer* $p(y|x)$, the conditional probabilities that $Y = y$ given $X = x$. Since, for all x and y,

$$p(x,y) = p(x)p(y|x),$$

and

$$p(y) = \sum_x p(x,y) = \sum_x p(x)p(y|x),$$

the required conditional probabilities that $X = x$ given $Y = y$ can be calculated by means of *Bayes' formula**

$$p(x|y) = \frac{p(x)p(y|x)}{\sum_x p(x)p(y|x)}.$$

An application of Bayes' formula will be found in the next section.

3.11 OPTIMAL ACCEPTANCE SAMPLING

The term *acceptance sampling* refers to sampling plans for controlling the quality of lots of manufactured or purchased items. Each item is assumed to be either Good (if it meets certain specifications) or Defective. These plans require that a random sample of items be selected from the lot, the number of defective items in the sample determined, and the lot as a whole rejected or accepted depending on whether or not the number of defectives in the sample is greater than a predetermined number (the "acceptance number"). Thus, the sampling plan is completely specified by two figures: the sample size and the acceptance number.

In this section, we shall show how the optimal sample size and acceptance number can be established. We shall do so in the context of a simple example, but the procedure can be easily generalized.

* The formula is named after Thomas Bayes (1702-61), English mathematician and clergyman, after whom an approach to statistical inference known as *Bayesian statistics* is also named.

Example 3.7 TRX Electronics is a manufacturer of high-fidelity equipment. Most of the components are made in-house, but some are purchased from other firms. One of these components is purchased from one supplier in lots of 10 items. Past experience with lots obtained from this supplier indicates the following probability distribution of the number of defective items in the lot:

Number of defectives in lot, x	Probability, $p(x)$
1	0.7
2	0.2
3	0.1
	1.0

For simplicity, we restrict x to a few (and rather unrealistic) values, but the procedure we are about to describe is perfectly general.

In order for TRX to be assured that no defective components are assembled into the final units, it will have to inspect every item in the lot and replace any defectives found with good items. The replacement items will be provided by the supplier, free of charge, but the cost of inspection must be borne by TRX. The cost of this inspection is \$45 per item inspected, or \$450 for the entire lot of 10 items. If the lot is not inspected, the defective components will cause the final units into which they are assembled to malfunction. The units will have to be dismantled, the defective components replaced, and the units reassembled. The cost of this operation is \$350 per defective final unit, or—since only one component goes into each final unit—per defective component.

Two possible alternatives, therefore, are open to TRX: (1) *full inspection*, i.e., inspect the entire lot before assembly; and (2) *no inspection*, i.e., accept the lot of components without inspection and replace any defectives when the final unit is tested.

The expected costs of the two alternatives, Full Inspection and No Inspection, are \$450 and \$490 respectively, as shown in Table 3.7.

The cost of Full Inspection is 10×45 or \$450 regardless of the number of defectives in the lot. If the number of defectives in the lot is 1, the cost of No Inspection is \$350; if there are 2 defectives in the lot, the cost of no inspection is 2×350 or \$700; finally, if there are 3 defectives in the lot, the cost is \$1,050. The expected cost of No Inspection is

$$(350)(0.7) + (700)(0.2) + (1050)(0.1) = 490.$$

Given, then, these costs and TRX's experience with lots of this component, the best act is to fully inspect the lot. The expected cost of the best

Table 3.7
Expected costs

Lot number defective, x	Probability, $p(x)$	Cost Full Inspection	No Inspection
1	0.7	450	350
2	0.2	450	700
3	0.1	450	1050
Expected cost:		450*	490

act is \$450.

There is, however, a third alternative: select and inspect a random *sample* of n components from the lot, count the number of defectives in the sample, and—depending on this number—either accept without inspection or inspect all remaining items in the lot.*

Actually, this is not one but a group of alternatives, one for each possible sample size. We shall first consider whether it is desirable to take a sample of a *specified* size. The same approach, however, applies to samples of any size.

Throughout this section we assume that sampling is without replacement.

Suppose that a sample is taken, and the number of defectives in the sample is determined. This number of defectives found serves as a piece of additional information. In order to evaluate the expected costs of the two alternatives at this stage (inspect all or none of the remaining items), we need the conditional probabilities of the lot number defective (x) given the observed number of defectives in the sample (y), $p(x|y)$. Neither $p(x|y)$ nor the joint probability that $X = x$ and $Y = y$, $p(x, y)$, are given. Note, however, that the marginal distribution of X, $p(X)$, *is* given, and the conditional distribution of Y given $X = x$, $p(Y|x)$, *can be inferred.* For $Pr(Y = y|X = x)$, the probability that in a random sample without replacement of size n there will be y defectives *given* that the number of defectives in the lot is x, is hypergeometric with parameters N, n and $k = x$. With known $p(x)$ and $p(y|x)$, the required $p(x|y)$ can be calculated by means of Bayes' formula.

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* Instead of inspecting all or none of the remaining items, another alternative would be to take a *second* sample, and, depending on the number of defectives found in this sample, to inspect all, inspect none of the remaining items, or take a *third* sample; and so on. These samples need not be of the same size. The problem of finding the optimal sampling plan becomes more complicated, but the method of solution is similar to that of this section.

To be specific, let us consider taking a sample of size $n = 3$. In a sample of 3 items there may be 0, 1, 2, or 3 defectives. Suppose that $Y = 1$ defective is found. The conditional probability that $X = x$ given $Y = 1$ can be calculated from Bayes' formula:

$$p(x|1) = \frac{p(x)p(1|x)}{\sum_x p(x)p(1|x)},$$

where $p(1|x)$ is the hypergeometric probability of 1 defective in a random sample of 3 items from a lot of size $N = 10$ containing x defectives. These conditional probabilities are calculated in column (3) of Table 3.8.

Table 3.8
Calculation of $p(x|1)$; sampling without replacement, $n = 3$

Lot number defective, x (1)	Probability, $p(x)$ (2)	$p(1\|x)^\dagger$ (3)	$p(x,1) = p(x)p(1\|x)$ (4)	$p(x\|1) = p(x,1)/\sum_x p(x)p(1\|x)$ (5)
1	0.7	0.3000	0.2100	0.5902
2	0.2	0.4667	0.0933	0.2623
3	0.1	0.5250	0.0525	0.1475
	1.0		0.3558	1.0000

$^\dagger p(1|x) = P_H(Y = 1|N = 10, n = 3, k = x)$

For example, if the lot contains 1 defective item, the probability that 1 defective will occur in a random sample of size $n = 3$ without replacement is 0.3000 (see Appendix 4J). The probability that the lot number defective will be 1 *and* 1 defective will be found in a sample of 3 items is (0.7)(0.3000) or 0.2100. The probability that the lot number defective is 1 *given* that 1 defective is found is 0.2100/0.3558 or 0.5902. The remaining conditional probabilities are similarly calculated.

Note that column (4) of Table 3.8 is one column of the joint probability distribution of X and Y, $p(x, y)$. This distribution could be shown as a table with three rows, corresponding to the possible values of X, and four columns, corresponding to the possible values of Y (0, 1, 2, 3). The sum of the entries in column (4), 0.3558, is the (marginal) probability that $Y = 1$.

Two acts are available: inspect all remaining components, and accept the lot without further inspection. The conditional expected costs of these acts are calculated in Table 3.9.

The cost of inspecting all remaining items in a lot of size N after a sample of size n is taken is:

Table 3.9
Conditional expected costs given $n = 3$ and $Y = 1$

Lot number defective, x	Conditional probability, $p(x\|1)$	Conditional cost Full Inspection	Conditional cost No Inspection
1	0.5902	315	0.00
2	0.2623	315	350.00
3	0.1475	315	700.00
Expected cost:		315	195.08*

$$(\text{Cost of full inspection}) = (N - n)(\text{Inspection cost per item}).$$

In this case, the cost of Full Inspection is $(10-3)(45)$ or \$315. The cost of No Inspection given that y defectives are found in a sample of n items without replacement from a lot of size N is

$$(\text{Cost of no inspection}) = \begin{cases} (x-y)(\text{Cost of replacing one defective}), & \text{if } y \leq x; \\ 0, & \text{if } y > x, \end{cases}$$

where x is the number of defectives in the lot. For example, if the number of defectives in the lot is 2, and if 1 defective is found in the sample, the cost of no inspection is $(2-1)(350)$ or 350. The expected cost of No Inspection is

$$(0)(0.5902) + (350)(0.2623) + (700)(0.1475) = 195.08.$$

Therefore, if 1 defective is found in the sample (and the probability of this is 0.3558), the best act is to accept the lot without further inspection. The expected cost of the best act is \$195.08.

Following exactly the same procedure, we calculate the conditional distribution of X, the marginal probability of Y, and the expected costs of the two acts for 0, 2, and 3 defectives. The reader can verify the summary of these calculations shown in Table 3.10.

If there are no defectives in the sample, the best act is Full Inspection. If 1 or more defectives are found, the best act is No Inspection.

If, therefore, a sample of size 3 is taken, the probability is 0.6125 that the expected cost will be \$315, 0.3558 that it will be \$195.08, 0.0308 that it will be \$198.65, and 0.0009 that it will be \$0. The overall expected cost of acting with the aid of a sample of size 3 is

$$(315)(0.6125) + (195.08)(0.3558) + (198.65)(0.0308) + (0)(0.0009) = 268.28.$$

These results are also illustrated in Figure 3.5. The expected cost of acting without the aid of this sample of size 3 is \$450; the expected cost

Table 3.10
Overall expected cost of acting with
the aid of sample of size 3 without replacement

Number defective, y	Proba-bility, $p(y)$	Conditional expected cost Full Inspection	No Inspection	Expected cost of best act
0	0.6125	315*	436.67	315.00
1	0.3558	315	195.08*	195.08
2	0.0308	315	198.65*	198.65
3	0.0009	315	0.00*	0.00
		Overall expected cost:		268.48

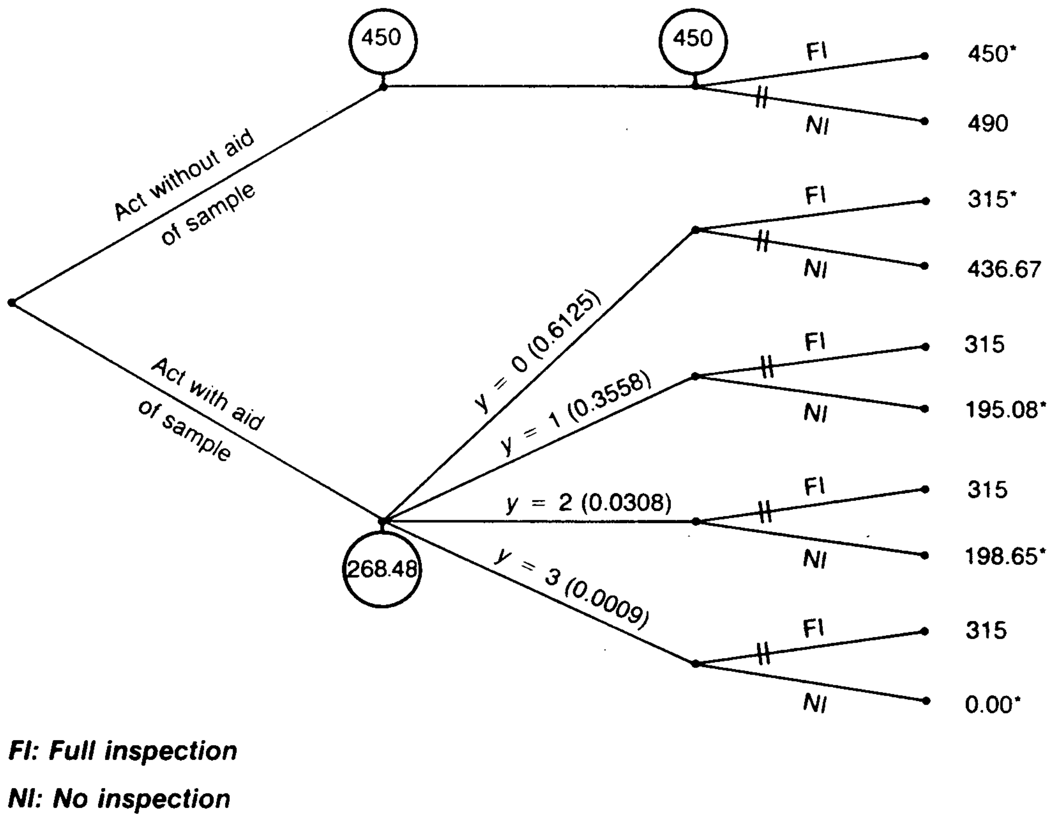

Figure 3.5
Sampling with the aid of sample of size 3 without replacement

of acting with the aid of the sample is \$268.48. *The expected value of the information provided by the sample* (EVSI) is the difference between these two quantities:

$$\text{EVSI} = 450 - 268.48 = 181.52.$$

The advantage of acting with the help of a sample of size 3 is partially offset, however, by the cost of taking this sample. Since the cost of inspection is \$45 per item inspected, the cost of taking the sample is \$135, and the *expected net gain from the sample* (ENGS) is

$$\text{ENGS} = (\text{EVSI}) - (\text{Cost of sample}) = 46.52.$$

Since the ENGS is positive, acting with the help of the sample of 3 items is preferable to not taking the sample.

Table 3.11
Expected net gain of sampling

Sample size, n	Overall exp. cost, OEC	Exp. value of sample information, EVSI[1]	Cost of sampling, SC[2]	Exp. net gain of sample, ENGS[3]
1	383.30	66.70	45	21.70
2	323.09	126.91	90	36.91
3	268.48	181.52	135	46.52
4	218.73	231.27	180	51.27
5	173.26	276.74	225	51.74*
6	131.63	318.37	270	48.37
7	93.55	356.45	315	41.45
8	57.96	392.04	360	32.04
9	25.70	424.30	405	19.30
10	0.00	450.00	450	0.00

[1]EVSI=450−OEC [2]SC=45n [3]ENGS=EVSI−SC

This does not necessarily mean that a sample of size 3 is optimal. Table 3.11 and Figure 3.6 show the overall expected cost, the expected value of the sample information, the cost, and the net gain for all other possible sample sizes. These quantities were calculated using a special computer program in exactly the same way as for a sample of size 3.

As the sample size increases, the EVSI also increases, but, in this case, at a decreasing rate. The cost of sampling is a linear function of the size of the sample. The expected net gain reaches a maximum of \$51.74 at $n = 5$. A sample of size 5, therefore, is optimal.

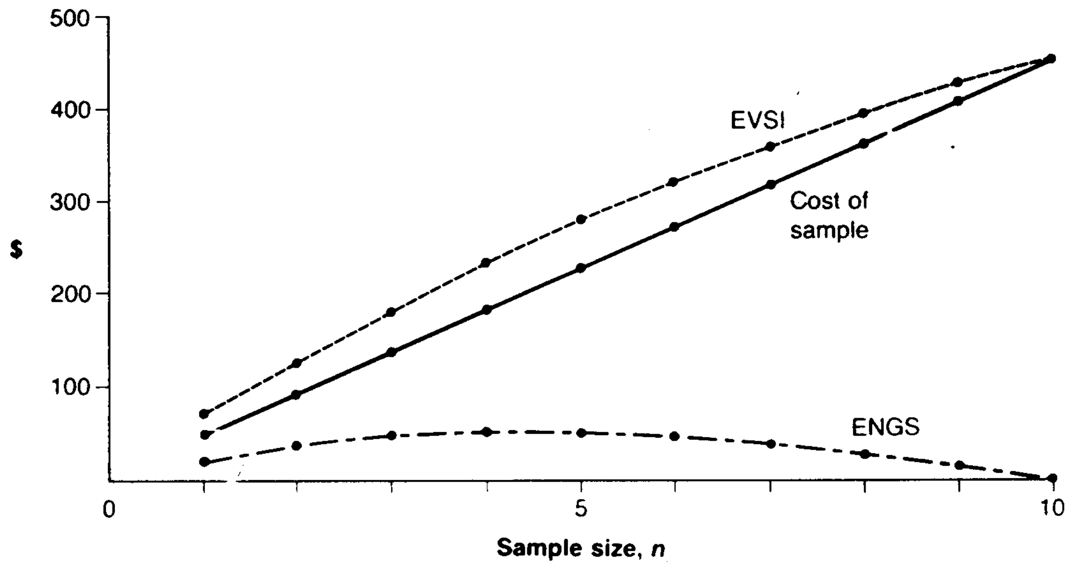

Figure 3.6
Optimal sample size, sampling without replacement

Table 3.12
Overall expected cost of acting with the aid
of a sample of size 5 without replacement

Number defective, y	Probability, $p(y)$	Best act	Expected cost of best act
0	0.4028	Full Inspection	225.00
1	0.5028	No Inspection	135.36
2	0.0861	No Inspection	169.36
3	0.0083	No Inspection	0.00
	1.0000	OEC:	173.26

Table 3.12 provides greater detail on the optimal sampling plan. This plan calls for taking a sample of 5 components without replacement, accepting the lot without further inspection if the number of defectives in the sample is 1 or more, and inspecting all components in the lot if the number of defectives in the sample is 0.

3.12 IN SUMMARY

The emphasis in this chapter has been on business problems in which the consequences of each alternative-event combination can be accurately expressed in dollars of profit, revenue, or cost.

A payoff table shows the available alternatives, the possible events and their probabilities, and the associated monetary consequences.

A reasonable (but by no means unique) criterion of choice is the maximization of expected profit or revenue, or the minimization of expected cost.

When alternatives and events succeed one another in stages, a decision tree is often useful in determining the best strategy. The procedure is to work backwards from last to first choices, calculating the expected payoff at each event node, and selecting the best alternative at each decision node.

The net value of an additional source of information is simply the difference between the best one can achieve using this information after subtracting the cost of obtaining it, and the best that can be achieved without the information.

PROBLEMS

3.1 The annual premium for a preferred homeowner's insurance policy, providing a "package" coverage of up to $50,000 on the house, $25,000 on the contents, and $100,000 for personal liability, with $*100 deductible*, is about $110. (Preferred rates apply essentially to houses less than 25 years old, insured to nearly their replacement value, with no more than one insurance claim in the past three years.) The "$100 deductible" means that if the amount of the claim is $$X$, the insurance company will pay $(X - 100)$ dollars if $X > 100$, and $0 otherwise. The premium of the same preferred insurance package but with $*50 deductible* is $120 per year.

For the purpose of this exercise, suppose that the relative frequency distribution of recent claims by homeowners is as follows:

Claim, $	Rel. frequ.	Claim, $	Rel. frequ.
0	0.30	60	0.05
10	0.05	70	0.05
20	0.05	80	0.05
30	0.05	90	0.05
40	0.05	100	0.05
50	0.05	More than 100	0.20
			1.00

Assume that all claims are reported regardless of the amount of the claim or the deductible.

You are the owner of a house and the package insurance coverage is exactly what you need. Should you buy the policy with $100 deductible or the policy with $50 deductible?

Show all calculations in the form of a payoff table. Explain why the distribution of past claims is or is not relevant to your decision.

3.2 A merchant buys a perishable product for \$$c$ per unit at the beginning of a day and sells it during the day for \$$p$. Any stock remaining unsold at the end of the day must be disposed of at the reduced price of \$$s$ per unit ($s < p$). If the demand is greater than the quantity available for sale, there is a "goodwill cost" of \$$g$ per unit of excess demand. Let $p(x)$ be the probability distribution of demand, X.

Show that the optimum order quantity, q^*, is the largest value of q for which

$$Pr(X < q) < \frac{p + g - c}{p + g - s}.$$

3.3 A merchant buys a certain product for \$0.89 and sells it for \$1.39. Any items unsold at the end of the period are disposed of at a price of \$0.39. The probability distribution of demand is as follows:

Demand, x	Probability, $p(x)$
50	0.15
60	0.25
70	0.20
80	0.15
90	0.10
100	0.08
110	0.07
	1.00

(a) Assuming that there is no goodwill loss associated with being out of stock, how many units should be purchased?

(b) If the goodwill loss is 0.20 per unit of unsatisfied demand, how many units should be purchased?

(c) The merchant has traditionally stocked 80 units. When shown the answer to (a) above, the merchant states that a goodwill cost for being out of stock has been incorporated. What is the implicit per unit goodwill cost associated with the inventory policy of 80 units?

3.4 In a given decision problem, the cost of act i is

$$c_i(x) = (x - a_i)^2 \quad (i = 1, 2, \ldots, n).$$

In words, the cost is a quadratic function of x, and a_i are known constants. Show that the optimum act is the one that minimizes $(\mu - a_i)^2$.

3.5 If the payoff of an act is given by

$$c(x) = a + bx + dx^2,$$

show that the expected payoff of the act is given by

$$E[c(x)] = a + bE(X) + d\{Var(X) + [E(X)]^2\},$$

where $E(X)$ and $Var(X)$ are the mean and variance of the probability distribution of X.

3.6 Verify the omitted calculations of Example 3.6.

3.7 Using the same data as in Example 3.6, calculate the overall expected profit of an ($s = 1$, $S = 2$) inventory policy if the inventory at the beginning of week 1 is (a) 1 unit; (b) 2 units; (c) 3 units.

3.8 Using the same data as in Example 3.6, calculate the overall expected profit of an ($s = 1$, $S = 1$) inventory policy if the inventory at the beginning of week 1 is 1.

3.9 A problem frequently faced by production departments concerns the number of items that must be scheduled for production in order to fill an order for a *specified* number of items. Each item may be good or defective, and there is uncertainty about the total number of good items in the production run. If too few items are produced, there may not be enough good items to fill the order; if too many items are produced, there may be more good items available than are required and the surplus items will be wasted.

As a simple illustration, consider the case of a production manager who has an order for 5 units of a particular product. The cost of setting up a production run is \$150 regardless of the size of the run. The variable manufacturing cost amounts to \$20 per unit produced. For a number of reasons, it is not possible to test each unit before the next unit is produced; rather, *all* the items in the run are tested at the same time, and the good items are separated from the defective ones at that stage. From past experience, the long-run fraction of defective units is estimated to be about 10%.

If there are fewer than 5 good units in the first run, a second run must be scheduled. The same comments apply for the second run as for the first. If the number of good units in the second run falls short of the number required, a third run may be necessary. Theoretically, a fourth, fifth, ... , run may be required if the number of good units in three runs still falls short of the requirements.

If more good units are produced in any one run than are required, the surplus good units are sold as scrap at \$10 per unit. Defective units are worthless.

(a) Under what conditions is it reasonable to suppose that the probability distribution of the number of good units in a run of n units is binomial with parameters n and $p = 0.90$? When answering the following questions, assume that these conditions are satisfied.

(b) In order to simplify the problem initially, assume that if a second run is required, it is possible to avoid having any defectives in that run by employing workers with more skill. The variable manufacturing cost in the second run will increase to \$40 per unit. How many units should be produced in the first run so that the expected cost per order is minimized?

(c) Suppose that if a second run is needed, it will be made under the same conditions as the first. However, if a *third* run is required, skilled workers will be employed, there will be no defective units in this run, and the variable cost will be \$40 per unit. Without doing any calculations, explain how you would determine the policy that minimizes the expected cost per order.

3.10 Vendex operates a downtown luncheon counter and can count on the patronage of a regular clientele from neighborhood offices. However, the fraction of customers showing up each day depends very much on the weather. Every morning Vendex must decide how much food to stock for noon. Let us suppose that the alternatives are:

a_1: Stock for the entire clientele
a_2: Stock for half the clientele
a_3: Close the shop

Vendex has calculated the following payoff table for this problem:

Payoff table
(In hundreds of dollars of profit)

	Events			
Acts	R&C	R&W	NR&C	NR&W
a_1	0	0	0	2
a_2	1	1	1	1
a_3	2	0	0	0

where R stands for rain, NR for no rain, C for cold, and W for warm weather. (Thus, for example, NR&C stands for no rain and cold weather.)

Two popular weather forecasts are given each morning. Since they are given at the same time, it is not possible to use both. The first station (A) specializes in forecasts of the type: "Warm weather today," and "Cold weather today." The second station (B) specializes in forecasts of the type: "Rain today," and "No rain today." Fortunately, both stations have kept records of the success of their predictions in the last 200 days:

Station A

	Actual weather			
Forecast	R&C	R&W	NR&C	NR&W
"Cold"	40	0	30	20
"Warm"	20	20	10	60

Station B

	Actual weather			
Forecast	R&C	R&W	NR&C	NR&W
"Rain"	50	10	10	20
"No Rain"	10	10	30	60

The tables show the number of days in which the indicated combinations occurred. For example, 40 in the upper left hand corner of the table for Station A indicates that in 40 of the last 200 days the station had predicted cold weather and the actual weather was rainy and cold.

Which station should Vendex rely on? Would Vendex be better off by ignoring both stations?

3.11 In a given decision problem, the profit (or cost) of each act is a function of the value of a random variable X. Another random variable, Y, could be used to improve decisions. Show that, if X and Y are independent, the expected value of the information provided by Y is zero.

3.12 Two processes (A and B) are available in the manufacture of a certain type of paper. Both processes convert raw material into the finished product. The raw material comes in two quality grades (Grade 1 and Grade 2). Batches of raw material come from various sources; it is impossible to determine their quality without the use of special tests. It is estimated, however, that about 60% of the batches are Grade 1 and 40% are Grade 2.

The current practice is to use process A on all batches of raw material regardless of their quality. The plant engineer recently suggested that a test be used to determine the quality of the raw material. The test costs \$30 per batch, and pronounces the batch as either "Class 1" or "Class 2." It is hoped that these two categories correspond to the two quality grades, but, as can be seen from the following table, the test results are not infallible:

	Test result		
Actual quality	"Class 1"	"Class 2"	Total
Grade 1	40	20	60
Grade 2	10	40	50

As indicated above, of 60 Grade 1 batches tested, 40 were pronounced "Class 1" and 20 "Class 2"; similarly, of the 50 Grade 2 batches tested, 10 were pronounced "Class 1" and 40 "Class 2."

The cost of manufacturing depends on the quality of raw material and the process used, as the following table shows:

	Cost per batch	
Raw material	Process A	Process B
Grade 1	\$100	\$160
Grade 2	\$150	\$120

Should the plant engineer's suggestion be adopted and the test used to examine all batches of raw material? Explain clearly, showing all calculations.

3.13 For some time now, the production manager of ABC Instruments was concerned about the high defective rate of one particular product. This product is used as a component of an aircraft instrument assembly and is sold exclusively by ABC at \$250 per unit. The component costs only \$150 per unit but ABC is able to charge a high mark-up since it has developed the component. According to the terms of the agreement between ABC and its clients, a penalty of \$50 will be paid by ABC for each defective component discovered by the client. In addition, ABC must provide a good component to replace the defective one. Defective components can be turned into good ones at an additional cost of \$20 per unit.

Despite efforts to control the quality of production, nearly 25% of all units (or 1 out of 4) are returned to ABC as defective. There seems to be some hope for the future, however. One of ABC's engineers has developed an instrument for testing the component with apparently good results. In a recent test, the instrument was applied to 100 known good components and 100 known defective components with the following results:

Instrument reading	Actual quality Good	Defective
"Good"	70	20
"Defective"	30	80
	100	100

Thus the instrument gave the correct reading on 70 of the known good and 80 of the known defective units.

It was suggested to the production manager that the test be adopted and all units tested before delivery. It was estimated that the tests would add about $10 to the unit cost of the component. All units identified as "Good" by the test would be shipped out as usual. All units identified as "Defective" would be reworked; the rework would cost $20 per unit, but reworked components would definitely be good.

Should the manager authorize the use of the instrument? How much is it worth to ABC?

3.14 Verify the calculations in Table 3.10 for 0, 2, and 3 defectives.

3.15 For the problem described in Example 3.7, verify the overall expected cost, the expected value of sample information, the cost of sampling, and the expected net gain for a sample without replacement of size: (a) 1; (b) 2; (c) 4; and (d) all other sizes shown in Table 3.11.

3.16 Consider Example 3.7, except that the distribution of the lot number defective, X, is:

x	$p(x)$
0	0.7
1	0.2
2	0.1
	1.0

Calculate the overall expected cost of a sample of size 2 without replacement. Use the table of hypergeometric probabilities shown in Appendix 4J.

3.17 The following table shows the probabilities of $Y = 0$, 1, 2 and 3 defectives in a random sample of $n = 3$ items drawn *without* replacement from a lot of $N = 20$ items of which x are defective. These are hypergeometric probabilities, and can be obtained from a table or special computer program.

Lot number defective, x	$P_H(Y = y \mid N = 20, n = 3, k = x)$ $Y = 0$	$Y = 1$	$Y = 2$	$Y = 3$
1	0.8500	0.1500	0.0000	0.0000
2	0.7158	0.2684	0.0158	0.0000
4	0.4912	0.4210	0.0842	0.0035

Assuming that the probabilities of the lot containing 1, 2, and 4 defectives are 0.3, 0.6, and 0.1 respectively, and that the costs of inspection and replacing a defective are the same as in Example 3.7, determine the overrall expected cost, the value of sample information, and the expected net gain of a sample of size $n = 3$ without replacement.