Direct proofs, proofs using contrapositive, proofs by contradiction

In addition to the proof of the following theorem, know the proofs to all of the examples in Section 1.7.

Theorem: There are an infinite number of prime numbers

In order to prove this we must call on the so-called **Fundamental Theorem of Arithmetic** which says that any integer may be written uniquely as a product of powers of powers of primes. This we will not prove but will take it as an axiom. It is not hard to convoke yourself that it must be true. Consider the example:

$$38 = 2^2 3^2 5^2$$

proof: Using a proof by contradiction assume the result is not true, namely there are only a finite number of primes. Assuming this we will work away and eventually end up with a contradiction. Then since each step of our work followed from the previous, the only thing wrong must have been the assumption that there were only finitely many primes.

This is how it goes - given that there are only finitely many primes, they may be listed as $p_1, p_2, p_3, \cdots, p_n$. Now form the number

$$X = p_1 p_2 p_3 \cdots p_n + 1$$

which is the product of all existing primes plus the number 1. Since X is itself not prime as it is greater than all primes $p_1, p_2, \cdots, p_n$, X must be divisible by some prime - say p_1. Thus $\frac{X}{p_1}$ is an integer. However, from below we see $\frac{X}{p_1}$ cannot be an integer , and thus there is a contradiction.

$$
\begin{aligned}
\frac{X}{p_1} &= \frac{p_1 p_2 p_3 \cdots p_n + 1}{p_1} \\
&= \frac{p_1 p_2 p_3 \cdots p_n}{p_1} + \frac{1}{p_n} \\
&= p_2 p_3 \cdots p_n + \frac{1}{p_n}
\end{aligned}
$$