1 Logic

Mathematics aims at trying to express ideas as clearly and unambiguously as possible. But this is not such an easy task - sometimes the ideas that we wish to express are complex, and it is easy for readers to become overwhelmed. Most people are unaccustomed to reading for precise meaning. We read novels and descriptive texts in which ambiguity is often part of the author's style. However for complicated texts the best way to minimize ambiguity is to use principles of logic - these rules are a tremendous aide to lazy people such as myself who do not wish to spend time meditating on what might otherwise be perceived as obscure writing. Using the ideas of logic simplifies life enormously. In these notes I give a very brief introduction - just enough so we can deal with some of the more abstract concepts. These notes then are very informal. The idea behind them is to introduce principles with a minimum of fuss.

What is meant by the word *logic?* In its broadest sense *logic* refers to mental patterns used thinking, and in this sense, logic is very much a cultural attribute. The related question of what is thought has occupied the efforts of most philosophers. Without attempting a definition we can however list characteristics - one being that *thought is a process of identifying and interpreting stimulus of the outside world.* The nature of the stimulus elicits particular responses. With repetition, the responses become patterns that grow more complex with time and firmly define parameters of action. People with different patterns interpret events in varying ways. Aboriginal peoples lived close to nature and were finely attuned to changes of season and weather. Their way of interpreting experience is worlds apart from that of the modern city dweller. But irrespective of differences there is always the problem of discerning meaning from events. This is a universal problem that runs across all cultures.

In the western tradition patterns of modern analytic thought may be traced back to the philosophic traditions of the ancient Ionians. Zeno (circa 450 B.C.) with the careful reasoning in his paradoxes shows understanding of the principles of logic. Only slightly later come the early contributors to what is now known as Euclidean geometry, Hippocrates of Chios, Archytas, Eudoxus, each of whom is credited with portions of Euclid's work. Also, in the writings of Plato and Aristotle one finds in the quality of the argument firm understanding of principles of logic. By this time what we consider as analytic logic was in common use.

An understanding and appreciation of the methods of logic seems to have arisen spontaneously in different cultures. There is a tradition in Tibetan Buddhism in which part of a monk's training consists of intense competition in logical debate. Similarly among Hebraic scholars there is an ancient tradition of logical analysis and debate with regard to interpretation of biblical text and various commentaries.

In this logic only data which is quantifiable in a direct sensory fashion is allowed for consideration. The data is subjected to detailed investigation resulting in certain statements and theories. Only theories arising from such an process are worthy of consideration. Today the notion of logic has been refined to a set of rules for manipulating statements and their meanings. These rules form a model for analytic thought often referred to as *Aristotelian or deductive logic*. But it is after all only a model - a simplified model that allows us to untangle meaning in certain circumstances.

The following exercises are meant to highlight the fact that in simple discourse problems of finding meaning can without some training can be puzzling.

Exercise 1 *A man was looking at a portrait in a castle in Scotland. The person standing next to him asked, "Whose picture are you looking at?" The man answered, "Brothers and sisters have I none, but this man's father is my father's son." Whose picture was he looking at?*

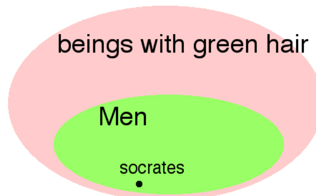
Exercise 2 *In a far away land there is a village in which the inhabitants are of two distinct types, knaves who always lie and knights who always tell the truth. One day while walking through the village I needed directions to the nearest post office and I approached 3 inhabitants of the village standing at a street corner. I have forgotten their names but lets call them A,B, and C. Of course to get a good answer I needed to know if whom I was speaking to was a knight or a knave. I approached A and asked,"Are you a knight or a knave?" He muttered something which I couldn't understand so I turned to B and asked," What did A say?" B replied, "A said that he is a knave." At this point the third person, C, said, "Don't believe B; he is lying!" The question then is, what are B and C - knights or knaves?*

Exercise 3 *In the same village suppose there are only two people, A and B. A makes the following statement: "At least one of us is a knave." What are A and B?*

Exercise 4 *Again in the same village suppose A says, "Either I am a knave or B is a knight." What are A and B?*

1.1 Arguments

If certain initial statements are assumed true deductive logic is the mental process by which conclusions are reached concerning the initial statements. The importance is that if the initial statements are considered true and there are no errors made in argument, then the conclusions should also be considered true. The difficulty comes in determining what makes a valid argument allowing one to get from assumptions to conclusion. As logic first developed the ability of determining a valid argument was a talent which came with practice in the analysis of simple arguments. Initial assumptions are called premises and an exercise consists of an argument consisting of stated premises and a conclusion. The reader is then asked to describe the logic by which the conclusion



was reached and the to classify the argument as valid or invalid. Following are some examples of what is meant.

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For the following **premises:**

(1) Socrates is a man and

(2) All men have green hair, we correctly deduce the conclusion:

Conclusion: Socrates had green hair.

Notice that not both of the premises are considered to be true. A good way to analyze the argument is to draw a *Venn diagram* to indicate the various categories mentioned in the premises

We argue from the picture that since the category of men is in the category of green-haired beings and since Socrates is in the category of men, it must follow that Socrates is also in the category of green-haired beings; so he himself has green hair. Thus the argument is valid, although we suspect that the conclusion is false. Suppose now that on the basis of some archeological evidence it is known that Socrates had black hair. Then substitute the word black for green in the argument. We then have now an example in which the premises are false, since not all beings have black hair, but the conclusion is true.

Now lets consider an example on an invalid argument.

Premises:

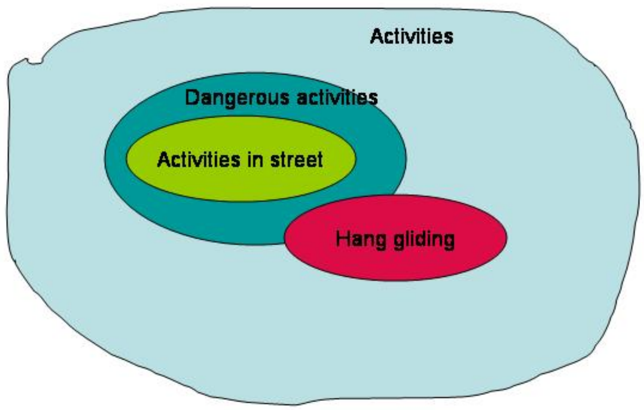
(1) All activities in the street are dangerous.

(2) Hang gliding is not done in the street.

Conclusion:

Hang gliding is not dangerous

In this example the premises would be considered true and the conclusion false. Lets analyze the argument with a Venn diagram



The first premise tells us that the category of activities in the street is contained in the category of dangerous activities which in turn is in the category of all activities. Premise 2 simply says that hang gliding is not an activity in the street. It therefore can be either dangerous or not dangerous as indicated by the diagram. Thus the logic giving the conclusion that hang gliding is dangerous is invalid.

The reasoning process by which one gets from a set of premises to a conclusion is the core of deductive logic. In the 19^{th} century this process was analyzed by the mathematician George Boole (1815-1864) and reduced to a study of symbolic operations. The following subsection provides a brief introduction.

Exercise 5 *State whether the following argument is valid or invalid. Give reasons*
Premises: All those who are drug addicts drank milk as a child. Joe drank milk as a child
Conclusion: Joe will become a drug addict

Exercise 6 *State whether the following argument is valid or invalid. Give reasons.*
Premises:If you do every problem in the book, then you will learn the subject. You learned the subject
Conclusion: You did every problem in the book.

Exercise 7 *State whether the following argument is valid or invalid. Give*

reasons.
Premises: If it snows today, the university will close. The university is not closed today
Conclusion: It did not snow today.

1.2 Propositions

Logic talks about *propositions.*

Definition 8 *A proposition is any sentence whose truth can always be determined*

Propositions are abstractly denoted with letters - we will for instance say, *let p be a proposition.* This means that the letter p stands for an arbitrary proposition. Given a proposition p, we also talk about its negation which we denote $\sim p$ - and which we call *not p.*

Definition 9 *Given a proposition p, its negation $\sim p$ is the proposition that is true whenever p is false and is false whenever p is true.*

In logic we only distinguish proposition from another by the conditions under which they are true. In fact we say that two propositions are *equivalent* if one is true precisely whenever the other is true. That is - two propositions may use different words in their construction, but we consider them the same if they have the same truth values.

Definition 10 *Given two propositions p and q. The proposition p is* ***equivalent to*** *the proposition q provided: (1) p is true whenever q is true and (2) p is false whenever q is false. We write $p \equiv q$.*

1.2.1 Combining propositions

There are several ways in which propositions can be combined to form new propositions. For instance if p stands for the proposition: "He has big feet" and q stands for the proposition: "He has green hair" , then *p and q* stands for the proposition: "He has big feet and he has green hair". Alternatively we write: $p \wedge q$ - where the symbol $\wedge$ means *and.* Similarly we can combine with the word *or.* For instance if p is the proposition: "the newest flu virus causes will cause a severe cough " and if q stands for the proposition: "the newest flu virus causes a high fever", then the proposition *p or q* stands for the proposition: "the newest flu virus causes either a severe cough or a high fever" . Alternatively we can write $p \vee q$, where the symbol $\vee$ stands for the word *or.* In logic the word *or* is used in such a way that the a proposition is considered to be true if both propositions are simultaneously true. Thus for in our example, *p or q* is true when p is true, when q is true, and when both p and q are true.

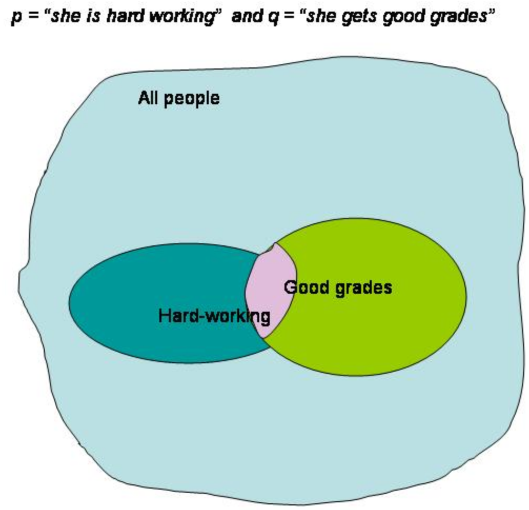


Figure 1: figure 3

In the following diagram, in which the propositions p and q stand for: $p =$ she is hard working and $q =$ she gets good grades the dark blue and pink regions stand for the category of all people who are hard-working and the pink and green regions stand for the category of all people that get good grades. Then the pink region stands for the category of people who are both hard working **and** get good grades; whereas the dark blue, pink, and green regions stand for the category of people who either are hard working or get good grades or both.

1.2.2 Forming the negation

Suppose now that we are given a compound statement such as: *she works hard and she gets good grades.* How do we form the negation of such a statement? One way is to consider Venn diagrams, and to argue from the picture. In general this is not a good method. Pictures get complicated and it is easy to make a mistake. But just for one last time, lets give it a try. We want to be able to describe in a simple way the negation of the proposition $p \wedge q$. In particular we want to describe the proposition $\sim (p \wedge q)$ in a more simple way in which the negation symbol does not stand in front of a more complicated expression. Consider diagram 3. If it is not true that a she works hard and gets good grades, then she can belong anywhere except in the pink region nor. That is if $\sim (p \wedge q)$ is true, she must lie outside the pink region. Now lets consider the case in which she does not work hard - namely $\sim p$ is true. She then belongs to either the green region or the light blue region. Similarly if she does not get good grades - namely $\sim q$, she then belongs to either the light or dark blue regions. All we need to do now is to realize that if she does not work hard **or** she does, not get good grades - namely the expression $\sim p \vee \sim q$ is true, then she belongs to the green, region or the dark blue region or the light blue region. What we have shown then is that each of the expressions $\sim (p \wedge q)$ and $\sim p \vee \sim q$ describe equally well which category our heroine might belong. We say that the two compound statements $\sim (p \wedge q)$ and $\sim p \vee \sim q$ are equivalent. We could run through a similar argument to show that expressions $\sim (p \vee q)$ and $\sim p \wedge \sim q$ are equivalent, but all of this is much easier if we first introduce truth tables.

1.2.3 Truth Tables

The truth table for a given compound statement involving statements p and q is a list of letters T or F indicating when the compound statement is true for various combinations of p and q being either true or false. The truth tables for $p \vee q$ and $p \wedge q$ are shown below. Note that $p \wedge q$ is true only when both components p and q are true, whereas $p \vee q$ is false only when both p and q are false.

p	q	$p \vee q$	$p \wedge q$
T	T	T	T
T	F	T	F
F	T	T	F
F	F	F	F

The truth table of the negation of a proposition is simply achieved by reversing $T's$ and $F's$. We then have the following

p	q	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$\sim (p \wedge q)$
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

and since the two expressions $\sim p \vee \sim q$ and $\sim (p \wedge q)$ can be shown to have the same truth values - one being true precisely when the other is - we say that the two expressions are equivalent. We write $\sim p \vee \sim q \equiv \sim (p \wedge q)$

1.2.4 The " if p, then q" statement

Most confusing however is the so called *if-then* combination - that is: given two propositions p and q, we can form the proposition *if* p, *then* q, which we write informally as $p \Rightarrow q$. Sometimes we also say p *implies* q - where the symbol $\Rightarrow$ stands for the word implies. The *if-then* statements is confusing because the statement is about causal connection and only secondarily does it involve the individual propositions. For an example, let p stand for the proposition, "it is raining", and let q stand for the statement "the street is wet". Then $p \Rightarrow q$ becomes the statement: "if it is raining, then the street is wet".

So what's the negation of this statement? Well - we need to express the negation of the causal connection between raining and the street becoming wet. We get the result: "it is raining **and** the street is not wet" Symbolically this is then: $p \wedge \sim q$ - or : " p and not q". In general then we have the following result.

Remark 11 *Given two propositions* p *and* q*, the negation of the proposition* $p \Rightarrow q$ *is the statement* $p \wedge \sim q$ *We write*

$$\sim (p \Rightarrow q) \equiv p \wedge \sim q$$

Now, given an *if-then* statement $p \Rightarrow q$, we can alter it as follows:

- $q \Rightarrow p$, the **converse** of $p \Rightarrow q$. In our example, this gives: "if the street is wet, then it is raining"
- $\sim q \Rightarrow \sim p$, the **contrapositive** of $p \Rightarrow q$, which gives : "if the street is not wet, then it is not raining"
- $\sim p \Rightarrow q$, the **inverse of** $p \Rightarrow q$, which gives: "if it is not raining, then the street is wet.

With a little bit of effort you may be able to convince your self of the fact that in general the statements $p \Rightarrow q$ and the contrapositive, $\sim q \Rightarrow \sim p$, are equivalent; that is, $p \Rightarrow q$ is true precisely when $\sim q \Rightarrow \sim p$ is true. This task is made immeasurably more easy by the consideration of truth tables. Below there are two tables, and between them is shown all 16 possible combinations of four of the two letters T or F. Each column corresponds to the truth values associated with the compound proposition at the head of the column. The column labeled $\mathfrak{T}$ stands for truth values of the statement what is known as the "universal affirmative" - namely any statement that is always true independent of p or q. Similarly the column labeled $\mathfrak{F}$ stands for the truth values of the statement known as the "universal negation" - a statement that is always false. All of the other columns can be obtained from the rules for determining whether $p \vee q$, $p \wedge q$, or $p \equiv q$ are true or false. As an exercise, make sure you know how each column is arrived at.

p	q	$\sim p$	$\sim q$	$p \vee q$	$p \wedge q$	$p \equiv q$	$\mathfrak{T}$	$\mathfrak{F}$
T	T	F	F	T	T	T	T	F
T	F	F	T	T	F	F	T	F
F	T	T	F	T	F	F	T	F
F	F	T	T	F	F	T	T	F

p	q	$p\wedge\sim q$	$\sim p\wedge q$	$\sim p\vee\sim q$	$\sim p\wedge\sim q$	$p\vee\sim q$	$\sim p\vee q$	$\sim(p\equiv q)$
T	T	F	F	F	F	T	T	F
T	F	T	F	T	F	T	F	T
F	T	F	T	T	F	F	T	T
F	F	F	F	T	T	T	T	F

One of the first things you might notice in examining these truth tables is that there is no column labeled $p\Rightarrow q$. What should we do? Examining $p\Rightarrow q$ we know that it must be true when both p and q are true, and it must be false when p is true and q is false. Now lets see which of the columns above begin with a T then an F. There are 3: $p\wedge q$, $p\equiv q$, and $\sim p\vee q$. Which should we assign to $p\Rightarrow q$. Since the first two are already taken up with well understood statements, we are left with assigning the truth values of $p\Rightarrow q$ to the column presently labeled $\sim p\vee q$. This however creates some initial confusion, for it tells us that the statement $p\Rightarrow q$ is always true whenever p is false. This would mean that the statement: "2+2 =5 implies the moon is made of blue cheese" is in fact a true statement. Culturally we are not accustomed to using the $p\Rightarrow q$ statement in the case that the first statement is false - that is in common speech we only use the first two rows of the truth table for $p\Rightarrow q$.

Exercise 12 *Prove using truth tables that* $p\Rightarrow q\equiv(\sim q\Rightarrow\ \sim p)$ *- namely a statement* $p\Rightarrow q$ *is equivalent to its contra-positive. Why does this also show that the inverse is equivalent to the converse?*

Exercise 13 *Prove using truth tables that* $\sim(p\vee q)\equiv\ \sim p\wedge\sim q$ *and* $\sim(p\wedge q)\equiv\ \sim p\vee\sim q$ *.*

Exercise 14 *Construct the converse, inverse and contrapositive of the following statements:*
(i) If the sun is shining, it will not snow in August
(ii) If the moon is made of blue cheese, Paul Martin is the prime minister.
(iii) If you do not eat, you will starve.

Exercise 15 *Construct truth tables for the propositions:*
(i) $(p\vee q)\wedge\sim p$
(ii) $(p\vee q)\vee\sim p$
(iii) $\sim(p\wedge q)\Rightarrow\sim(p\vee q)$
(iv) $q\Rightarrow(p\vee q)$
(v) $(p\Rightarrow q)\wedge(q\Rightarrow p)$

1.3 Quantifiers

Certain propositions are more complicated than others - even before they are combined in the ways talked about above. Some propositions make reference to a large group of objects- sometimes called the *universe of discourse*. Consider the proposition, “ all politicians are dishonest”. This is statement about an entire group of people. Consider also - " there exists at least one politician that is not dishonest" . Again we are referring to the entire category of politicians. Such statements are intrinsically more complicated than say, "the street is wet", and are examples of quantified statements. These examples can be rephrased slightly in a way that will allow further analysis. For instance, in the case that the universe of discourse is the set of all people, we can rephrase, " all politicians are dishonest" as

- "for every person x, if x is a politician, then x is dishonest".

The other, "there exists at least one politician that is not a dishonest" can be rephrased as:

- " there exists a person x such that x is a politician and x is not a dishonest".

It is also evident that the second statement above is the negation of the first

Now with this example, lets see how we might generalize. As before let p and q be arbitrary propositions - say p is the proposition " x is a politician" and q is the proposition "x is dishonest". Then letting the symbol $\forall$ stand for the words *for every* and the symbol $\exists$ stand for the words *there exists,* and letting U stand for the set of all people then the above quantified statements can be expressed respectively as

1. $\forall x$, in U, $p \Rightarrow q$ and
2. $\exists$ x in U such that $p \wedge \sim q$

Considering the original examples, it is clear that the second is the negation of the first. Symbolically, using the symbol $\in$ to mean "is an element of" we can then write:

$$\sim (\forall x \in U,\ p \Rightarrow q) \equiv \exists\ x \text{ such that } p \wedge \sim q$$

In more general terms, what we have said here is that to negate a statement with the quantifier $\forall$, change $\forall$ to $\exists$ and then negate the following statement. That is $\sim (\forall x,\ p \Rightarrow q)$ is equivalent to: $\exists$ x such that $\sim (p \Rightarrow q)$ $-$ which in turn is equivalent to: $\exists$ x such that $p \wedge \sim q$. All of this is true in yet more generality. Given a statement made up of a string of quantifiers and a final statement. The rule for negating such a statement runs as follows.

Remark 16 *To negate a propositions constructed from a string of quantifiers $\exists$ and $\forall$ followed by a final statement p, do the following: (1) change every occurrence of $\forall$ to $\exists$, (2) change every occurrence of $\exists$ to $\forall$, (3) negate the proposition p, (4) adjust the use of language so the sentence reads nicely.*

Exercise 17 *Let the universe of discourse U be the collection all students. For $x \in U$ and $y \in U$, let $C(x)$ stand for the proposition:* x has a computer *and let $F(x, y)$ stand for the proposition:* x and y are friends. *Translate the following statement*

$$\forall x \; C(x) \vee \exists y(C(y) \wedge F(x, y)$$

noting that sometimes one needs to insert a few words to make the sentence readable

In the above exercise observe that if one were to place all quantifiers at the beginning of the expression, we do not alter the meaning. Thus the expression in the exercise could have been written

$$\forall x \exists y \;\; C(x) \vee (C(y) \wedge F(x, y)$$

Exercise 18 *Translate the statement*

$$\exists x \forall y \forall z \; (((F(x, y) \wedge F(x, z) \wedge (y \neq z)) \Rightarrow \;\; \sim F(y, z))$$

where the universe of discourse and F are as in the previous exercise.

Exercise 19 *In the previous two exercises form the negations both of the symbolic expressions and their English equivalents*

With the tools so far we can now express arguments symbolically. Consider the following example

Example 20 *Premises:*
All lions are fierce.
Some lions do not drink coffee.
Conclusion:
Some fierce creatures do not drink coffee

In the example let the universe of discourse be the set of all "creatures". Let $P(x), Q(x), R(x)$ be the statements: *x is a lion, x is fierce, and x drinks coffee.* We can then rewrite the argument as:

$$\begin{array}{l} \forall x \, (P(x) \Rightarrow Q(x)) \\ \exists x(P(x) \wedge \sim R(x)) \\ \exists x(Q(x) \wedge \sim R(x)) \end{array}$$

Lets take a close look at the second premise as we have expressed it symbolically. Could we instead have written $\exists x(P(x) \Rightarrow \sim R(x))$. Observe however that $P(x) \Rightarrow \sim R(x)$ is true whenever x is not a lion. Thus the entire statement is true as long as there is one creature that is not a lion - even if every lion drinks coffee. With the same reasoning, the third statement cannot be written $\exists x(Q(x) \Rightarrow \sim R(x))$.

Exercise 21 *Consider the following argument:*
Premises:
All hummingbirds are richly colored
No large birds live on honey.
Birds that do not live on honey are dull in color
Conclusion:
Hummingbirds are small
Let $P(x), Q(x), R(x), S(x)$ *be the statements:* x *is a hummingbird,* x *is large,* x *lives on hone,* x *is richly colored. Assuming that the universe of discourse is the set of all birds, express the argument symbolically as in the previous example.*

1.3.1 More truth tables

What I have briefly described in the preceding paragraphs is otherwise known as the *prepositional calculus.* By this I mean the rules that allow one to formally manipulate statements. The truth tables that were developed assumed that there were only always two propositions p and q that were under discussion. This of course is not always the case, but using the same principles truth tables can be constructed for any number of propositions. Suppose for instance that we are considering statements involving 3 propositions, p, q, and r. Considering all the possibilities for each proposition being either true or false, we would then need to examine columns of $T's$ and $F's$ of length 8 such as in the following table

p	q	r	$(p \vee q) \wedge r$	$(p \wedge q) \vee r$
T	T	T	?	?
T	F	T	?	?
F	T	T	?	?
F	F	T	?	?
T	T	F	?	?
T	F	F	?	?
F	T	F	?	?
F	F	F	?	?

2 Brief epilogue

Although the principles of logic had been known for millennia, the formulation as the propositional calculus came about only in the latter part of the 19^{th} century, as part of the attempt at that time to put the foundations of mathematics and hence all scientific discourse on a more firm foundation. By that time the fabric of mathematics had out grown the Euclidean foundations. In particular the discovery of the possibility of non-euclidean geometry presented a dilemma. Interestingly enough the foundations in all the years had not yet deeply examined. The goal of the 19^{th} century logicians was to place the foundations of mathematics on logic - that is: replasce the Euclidean axions and postulates with principles built bit by bit only from principles of logic. What could be more fundamental? This was a grand and ambitious programme to which many prominent mathematicians, logicians, and philosophers devoted their careers. The formulation of logic in terms of the propositional calculus is the work of George Boole (1815-1864) an English mathematician. Following is a quote from another 19th century English mathematician Augustus De Morgan

Boole's system of logic is but one of many proofs of genius and patience combined. ... That the symbolic processes of algebra, invented as tools of numerical calculation, should be competent to express every act of thought, and to furnish the grammar and dictionary of an all-containing system of logic, would not have been believed until it was proved. When Hobbes ... published his "Computation or Logique" he had a remote glimpse of some of the points which are placed in the light of day by Mr Boole.

Boole's work was taken up by others but the grand scheme of placing mathematics on the foundation of logic came across a number of bumps in the road. In particular there is the so called "Russell paradox" that was constructed by then a young and brash Englishman of the name Bertrand Russell in (1901). There followed in 1931 the so called "Gödel paradox" due to the Czech-born mathematician Kurt Gödel. Today the work on foundations continues.

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3 Real Numbers

3.1 Taxonomy

The set of real numbers or all the points on a straight line consist of

- the **natural numbers** - or counting numbers - namely $\{1, 2, 3, 4, \cdots\cdots\}$ which are denoted with the symbol $\mathbb{N}$

- the **integers** - otherwise known as whole numbers - namely the set $\{\cdots,$ $-3\ -2, -1,\ 0,\ 1\ ,2\ ,3, \cdots\}$ which is denoted by the symbol $\mathbb{Z}$

- the **rational numbers** - namely the set of all fractions in lowest terms where fractions of the form $\frac{p}{1}$ are identified with the integer p. This set is denoted with the symbol $\mathbb{Q}$

- the **irrational numbers** - namely all those numbers that are not fractions. Among the irrationals there are those that are the roots of equations such as $\sqrt{2}$ and those which are not the root of any equation such as π - the latter are called **transcendental numbers.**

  The entire set of real numbers itself is denoted with the symbol $\mathbb{R}$.

3.2 A model for the real numbers

The set of real numbers can be identified with the collection of all equivalent infinite decimal expressions. That is the number 1 is identified with the infinite decimal expression $1.00000\cdots = 0.999999999999\cdots$, the number $\frac{1}{2}$ is identified with the infinite decimal expression $.50000000\cdots = .49999\cdots$. It is fairly obvious that the normal procedure of converting a fraction into a decimal by dividing the denominator into the numerator allows only a finite number of remainders at any stage and that thus eventually a remainder will repeat. When this happens an entire block of digits in the quotient begins to repeat. Try dividing 7 into 2 - you will see what I mean.

The interesting fact is that the converse is also true. That is, given an infinite decimal representation that eventually repeats some pattern, this decimal is the representation of some rational number. Here is how it works. We will illustrate this fact by example. Suppose we have the infinite decimal $x = 1.24523232323\cdots$ which eventually repeats infinitely the block of digits 23. Multiply x by 10^n where n is the length of the repeating block. In this case we multiply by 10^2 getting $100x = 124.523232323\cdots$. Now subtract x from $100x$

$$\begin{array}{rr} & 124.523232323\cdots \\ - & \underline{1.24523232323\cdots} \\ & 123.2780000000\cdots \end{array}$$

or in other words $100x - x = x(99) = \frac{123278}{1000}$ or $x = \frac{123278\times 99}{1000} = \frac{6102\,261}{500}$ which is a rational number. We now can state the following result.

Proposition 22 *Every rational number, which includes integers and natural numbers, has a representation as an infinite decimal expression that eventually repeats a certain fixed pattern of digits. Conversely, any infinite decimal expression that eventually repeats a certain fixed pattern of digits is the representation of some rational number.*

With this proposition at our disposal it is easy to lay hands on irrational numbers. They are simply those numbers which **do not** have an eventually repeating decimal expression. Such a decimal expression is $1.01001000100001000001\cdots$ where the block of zeros grows in length indefinitely.