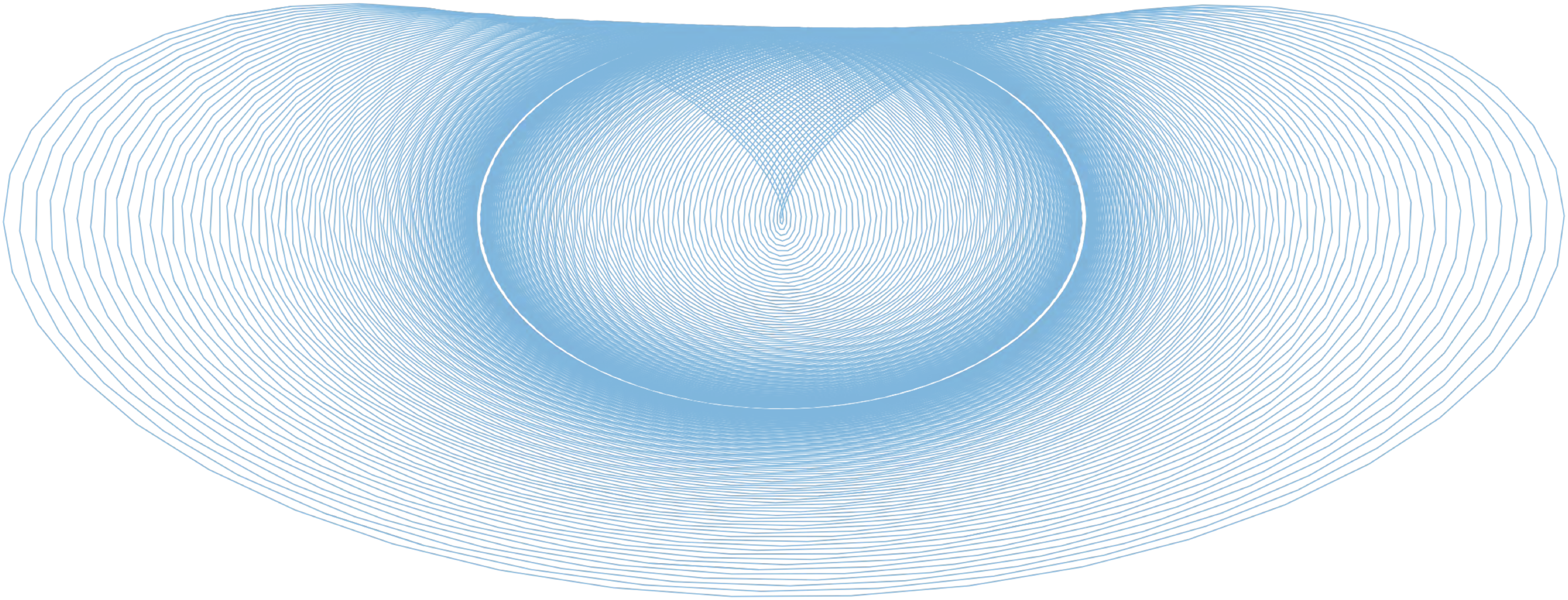
PHYS 1420 (F19)
Physics with Applications to Life Sciences



2019.11.01
Relevant reading:
Kesten & Tauck ch. 8.2-8.5

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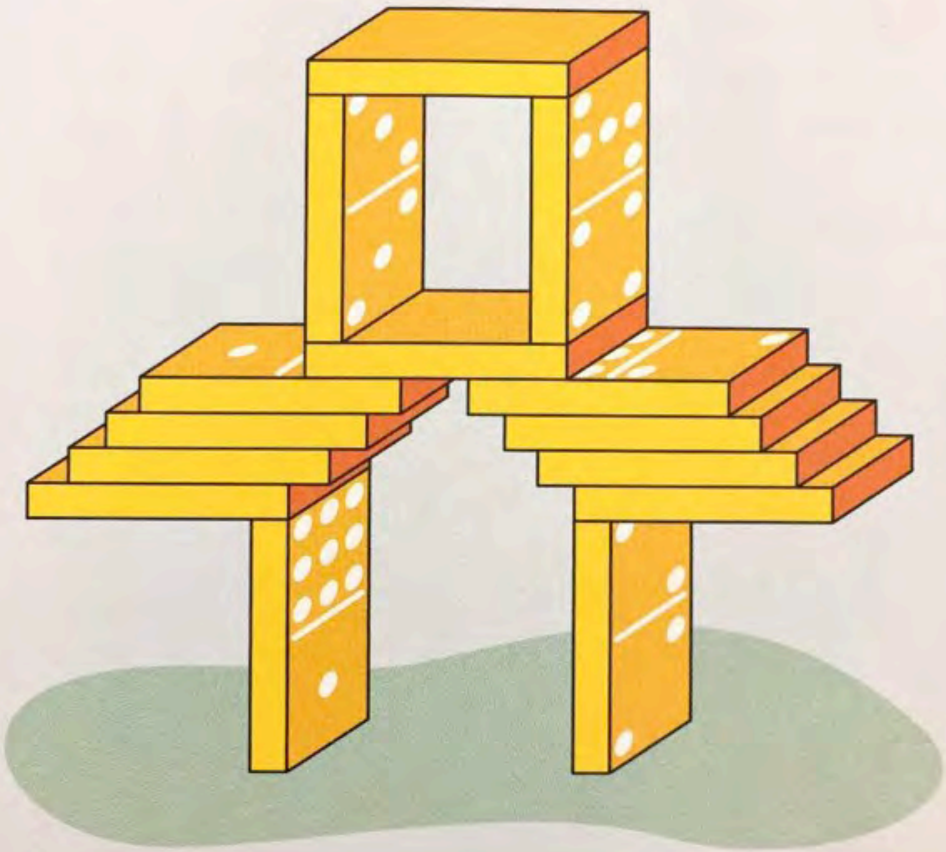
Ref. (re images):
Wolfson (2007), Knight (2017),
M. George, Kesten & Tauck (2012)



The impossible domino bridge problem

At first sight the structure above is impossible to build. But if you think about it the right way, you can work out how to do it and even build it yourself!

→ This is a practical example of a center-of-mass calculation!

Announcements & Key Concepts (re Today)

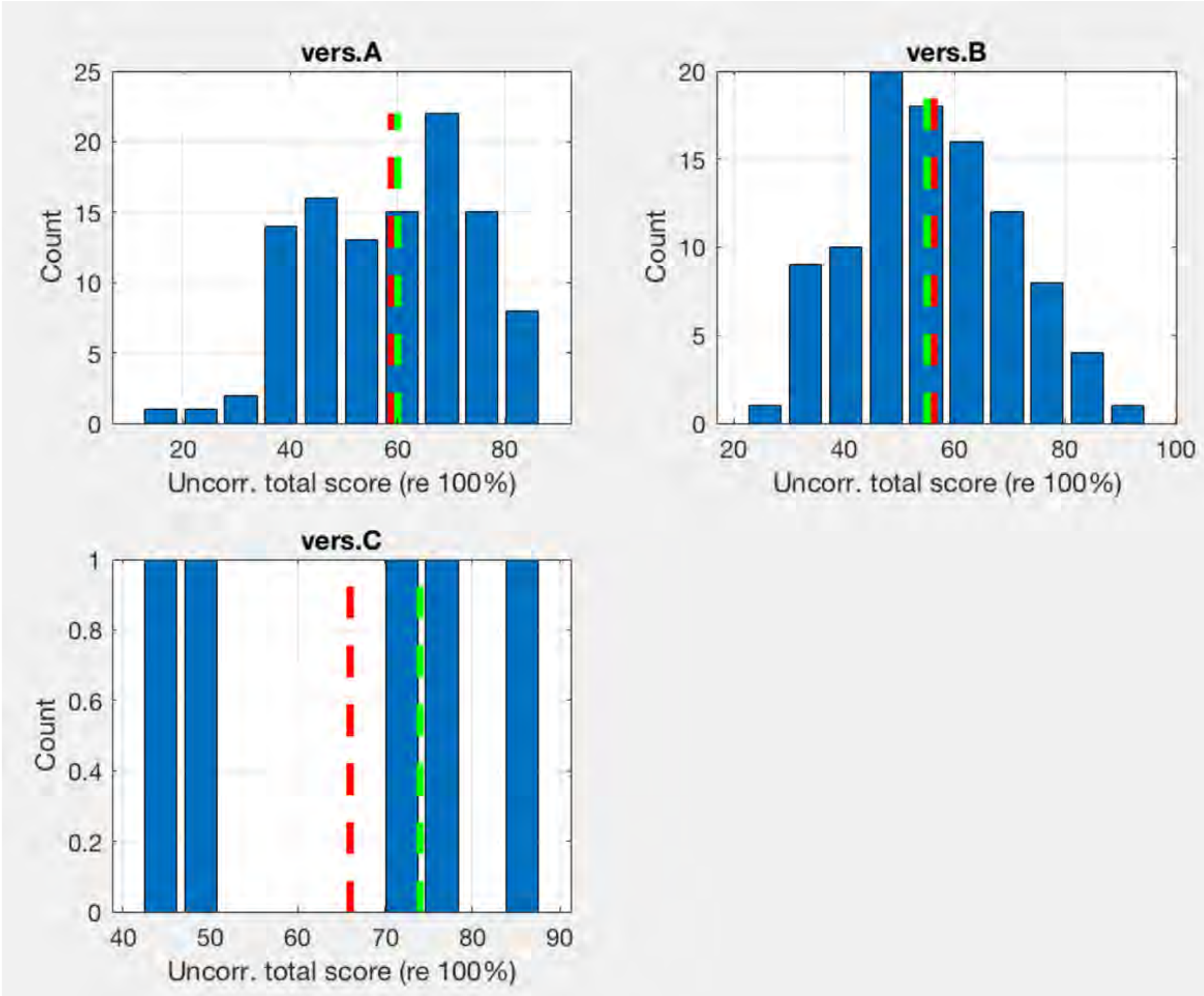
→ Online HW #7: Posted and due Monday Nov. 4

→ Midterm exams are graded and will be handed back next week

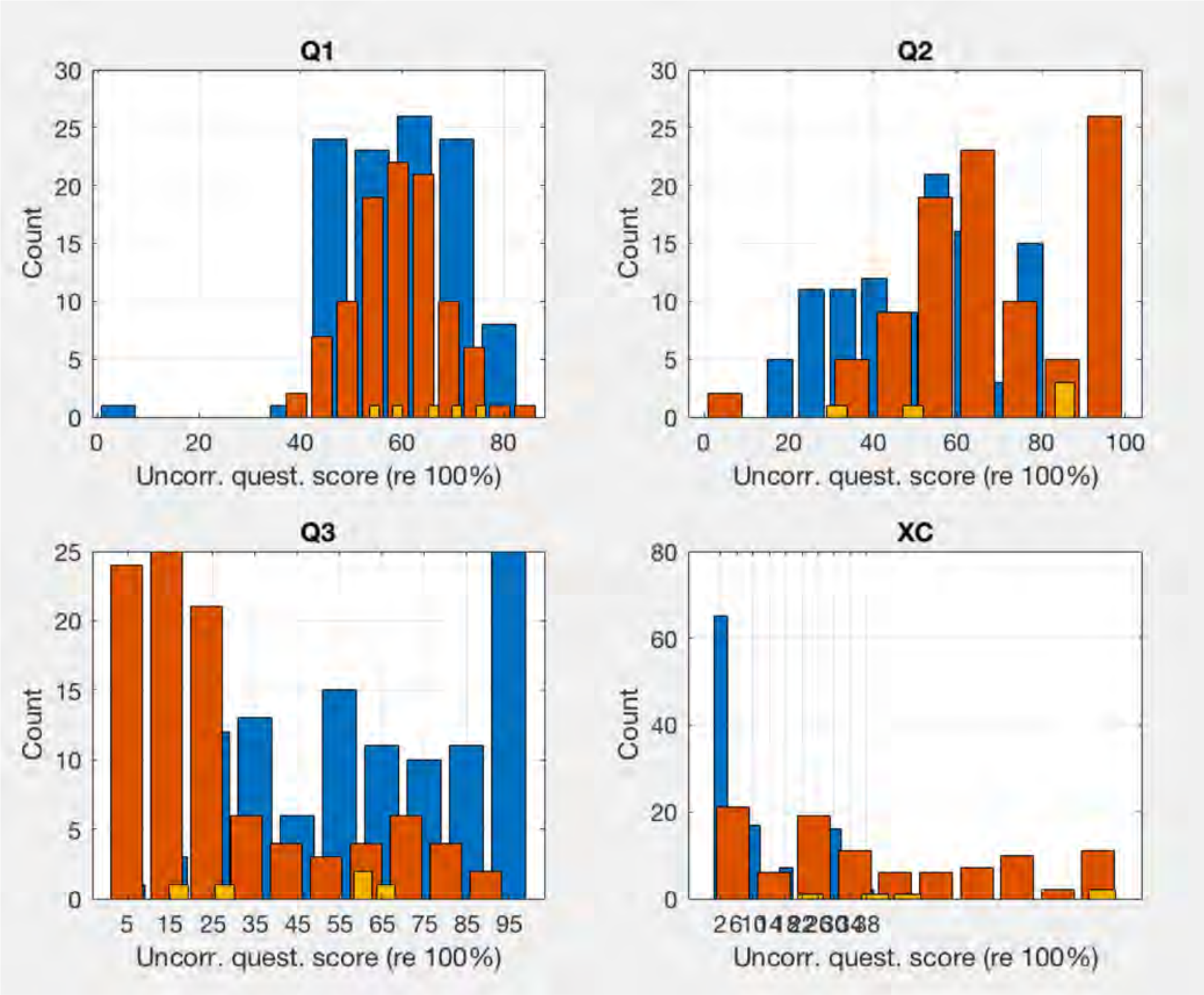
Some relevant underlying concepts of the day…

- Moment of inertia
- Parallel-axis Theorem
- Rotation + Conservation of Energy

Preliminary midterm scores....

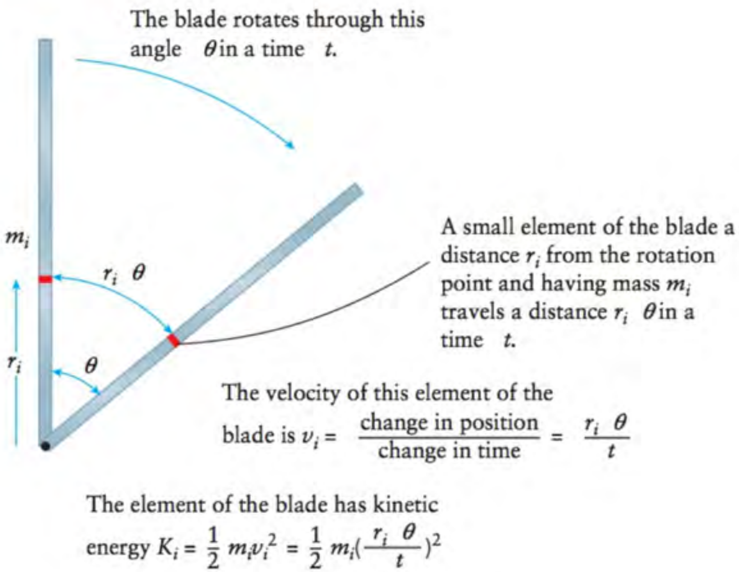


Preliminary midterm scores....



Blue is vers.A
Orange is vers.B

Review: Rotational Kinetic Energy



$$K = \frac{1}{2}\left(\sum m_i r_i^2\right)\omega^2$$

Moment of inertia

$$I = \sum m_i r_i^2$$

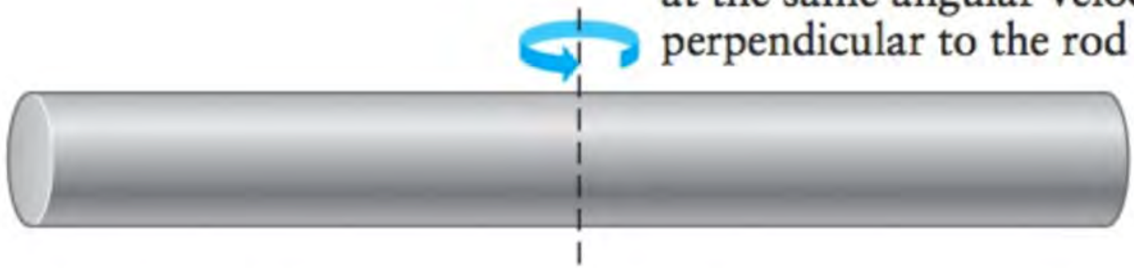
$$K_{\text{rotational}} = \frac{1}{2} I \omega^2$$

Whereas we interpreted mass as a property of matter that represents the resistance of an object to a change in translational velocity, the moment of inertia represents the resistance of an object to a change in rotational or angular velocity. In the same way that we defined inertia as the tendency of an object to resist a change in translational motion, we can define rotational inertia as the tendency of an object to resist a change in rotational motion.

Rotational Kinetic Energy

The kinetic energy of a rod rotating around an axis perpendicular to the rod and through its end is four times larger...

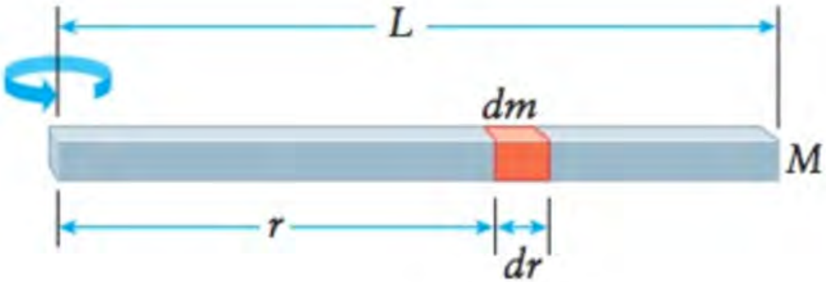
...than the kinetic energy of the rod when it rotates at the same angular velocity around an axis perpendicular to the rod and through its center.

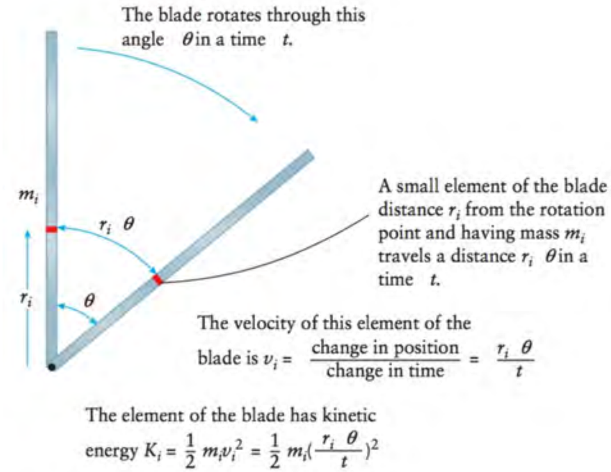


The more mass there is farther from the rotation axis, the larger the moment of inertia, and the larger the rotational kinetic energy for any given angular velocity.

Kesten & Tauck

Moment of Inertia: Uniform bar rotating about one end





Moment of inertia

$$I = \sum m_i r_i^2$$

Now in integral form, but...

$$I = \int r^2\, dm$$

... how are r and m functionally-related?

$$I = \int r^2 \frac{M}{L} dr = \frac{M}{L}\int r^2\, dr$$

Uniform density!
(equal proportions)

$\Delta m/\Delta r = M/L$

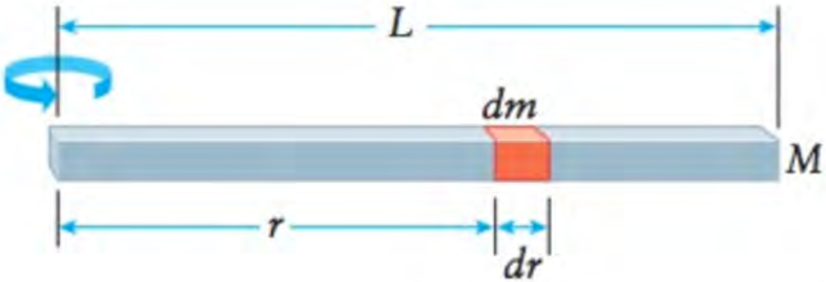
Now in the limit:

$$dm = \frac{M}{L} dr$$

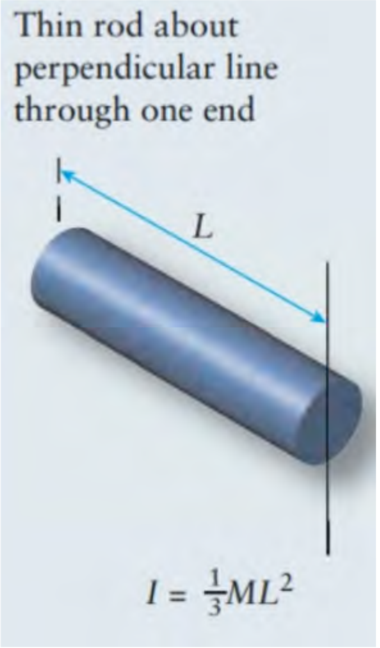
$$I = \frac{M}{L}\frac{r^3}{3}\Big|_0^L = \frac{M}{L}\left(\frac{L^3}{3} - \frac{0^3}{3}\right) = \frac{ML^2}{3}$$

Kesten & Tauck

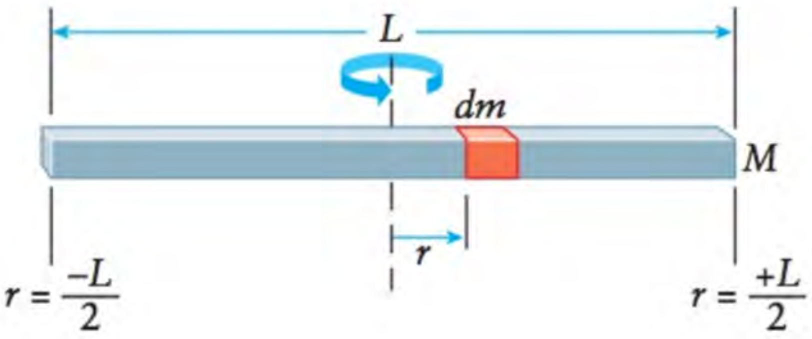
Moment of Inertia: Uniform bar rotating about one end



$$I = \frac{M}{L}\frac{r^3}{3}\Bigg|_0^L = \frac{M}{L}\left(\frac{L^3}{3} - \frac{0^3}{3}\right) = \frac{ML^2}{3}$$



Moment of Inertia: Uniform bar rotating about its center



An object does not have "a" moment of inertia. Rather, it has a moment of inertia defined for rotation around *each specific choice* of rotation axis.

$$I = \frac{M}{L}\int_{-L/2}^{+L/2} r^2\, dr$$

Just change the limits of integration!

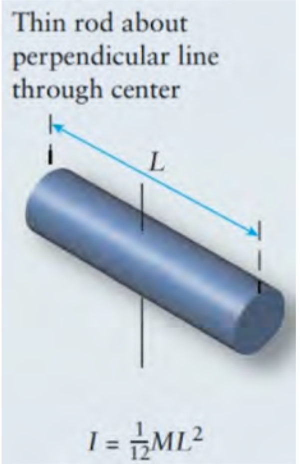
$$= \frac{M}{L}\frac{r^3}{3}\Big|_{-L/2}^{+L/2} = \frac{M}{L}\left(\frac{(+L/2)^3}{3} - \frac{(-L/2)^3}{3}\right)$$

$$= \frac{M}{L}\left(\frac{L^3}{24} - \frac{-L^3}{24}\right) = \frac{ML^2}{12}$$

Recall: Note the similarity here w/ respect to CM calculations...

$$x_{CM} = \frac{1}{M_{tot}}\sum_{i=1}^{N} m_i x_i$$

... a **key difference being that here things are moving** (or more specifically, *rotating*)

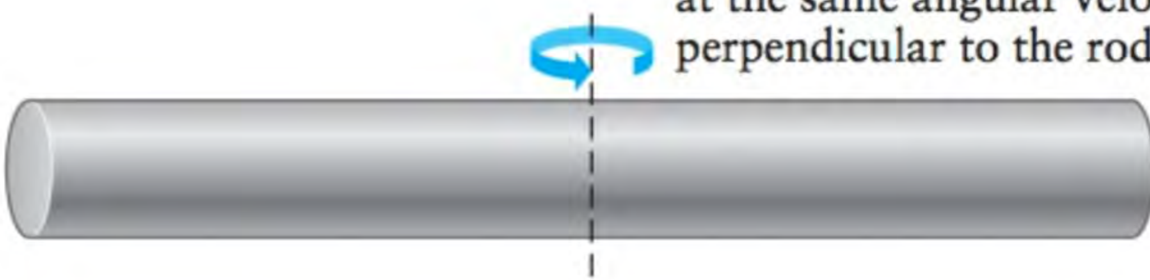


Question: What is a *thin rod* and why is that needed?

Kesten & Tauck

The kinetic energy of a rod rotating around an axis perpendicular to the rod and through its end is four times larger...

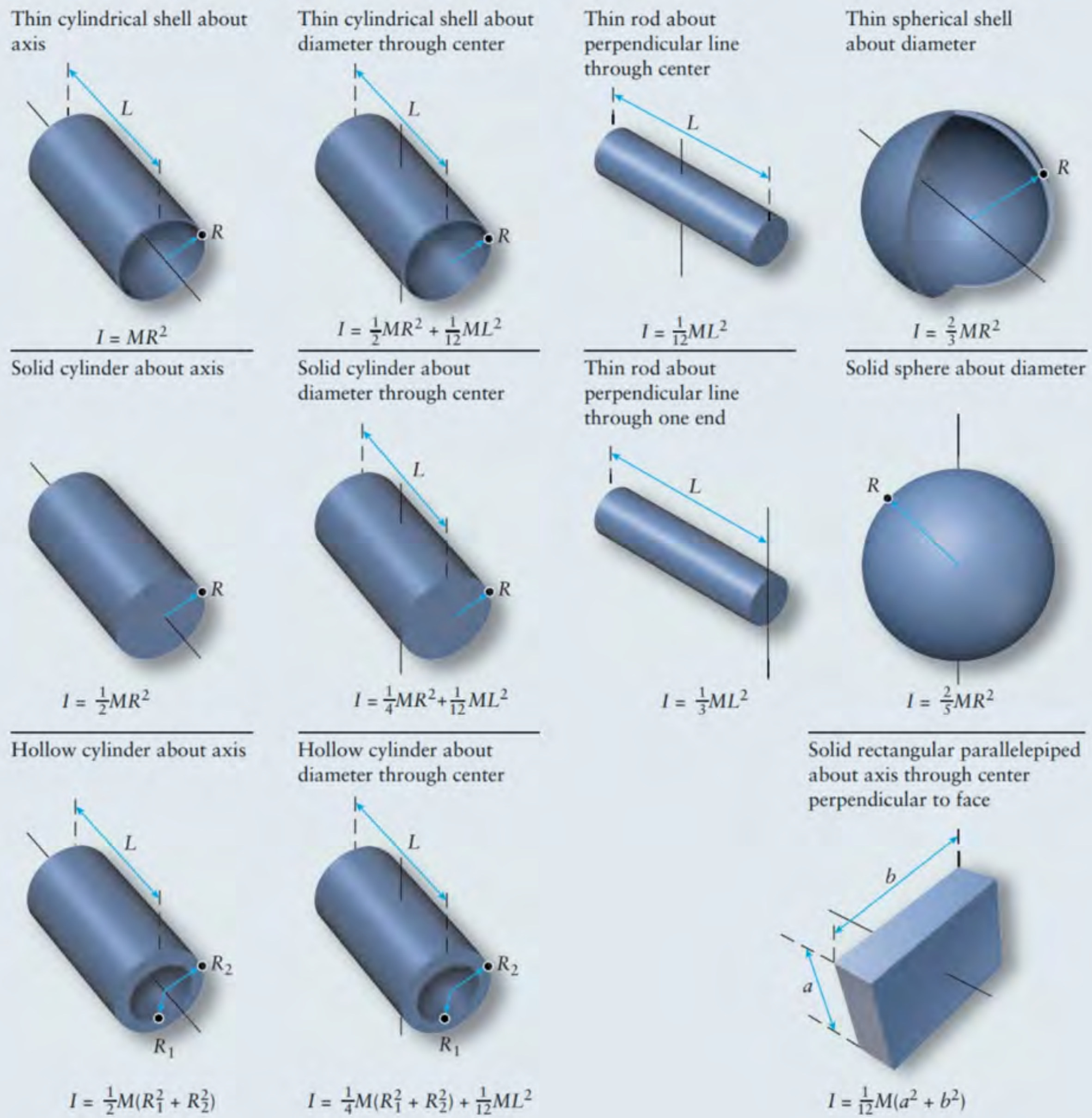
...than the kinetic energy of the rod when it rotates at the same angular velocity around an axis perpendicular to the rod and through its center.



The more mass there is farther from the rotation axis, the larger the moment of inertia, and the larger the rotational kinetic energy for any given angular velocity.

Kesten & Tauck

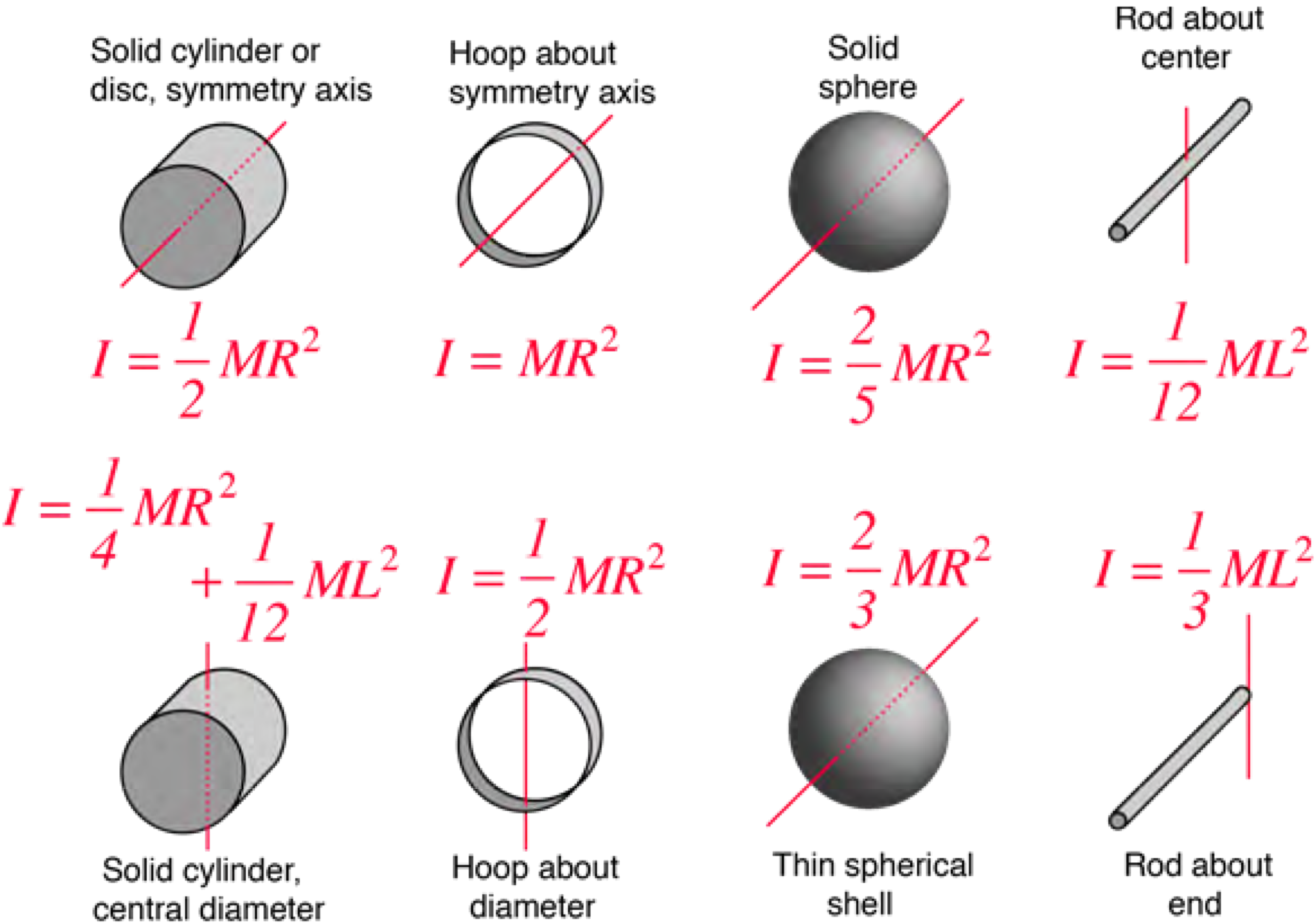
Moments of Inertia of Uniform Bodies of Various Shapes*



*A disk is a cylinder whose length L is negligible. By setting $L = 0$, the above formulas for cylinders hold for disks.

Kesten & Tauck

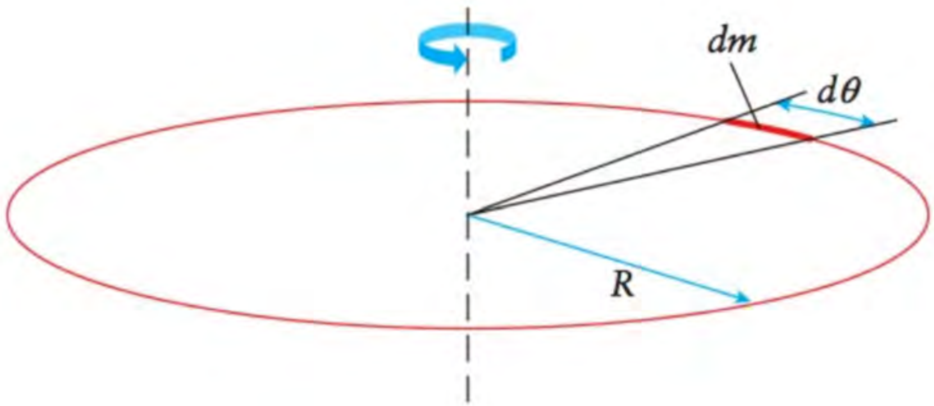
MomentS of Inertia

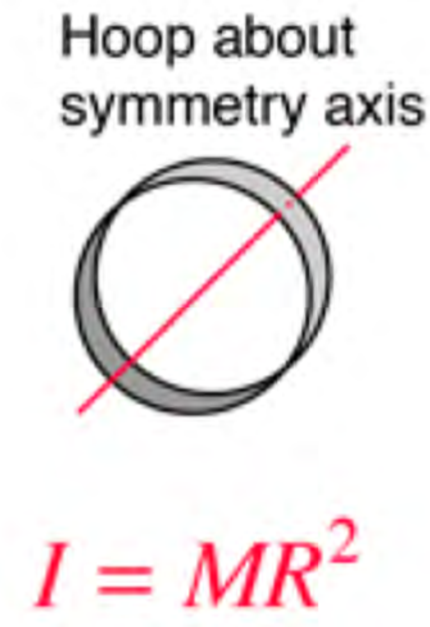


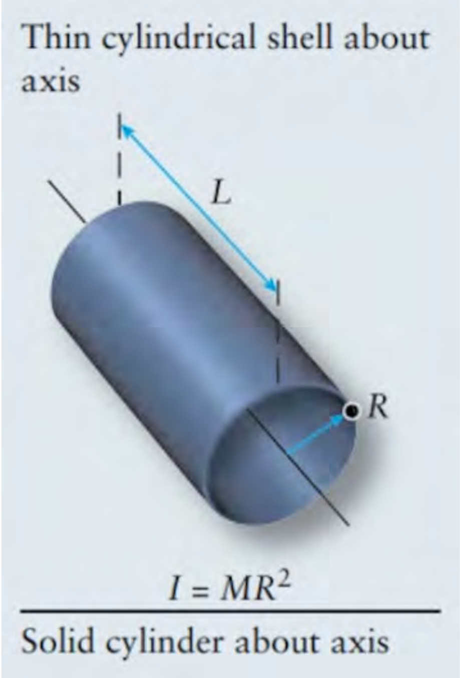
M. George

<u>Aside</u>: Some of those other shapes....

Thin hoop rotating about center



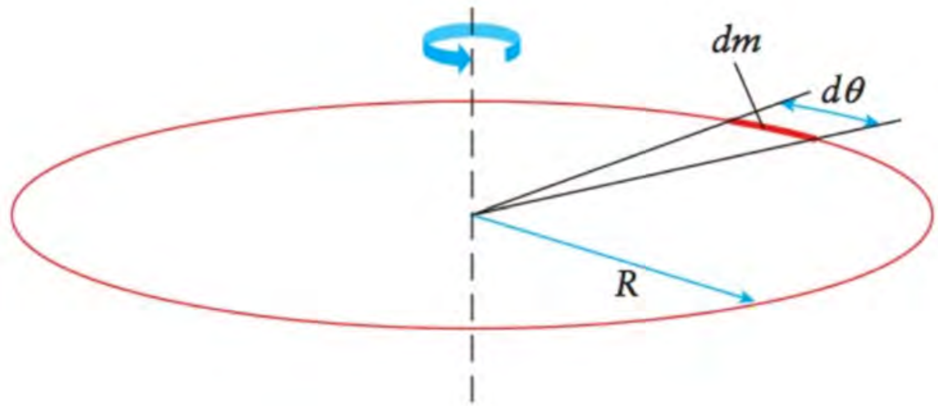




→ When does a "thin hoop" become a "thin cylindrical shell?

<u>Aside</u>: Some of those other shapes....

Thin hoop rotating about center



Similar argument before re density & proportionality

$$\frac{\text{angle of slice}}{\text{angle of ring}} = \frac{\text{mass of slice}}{\text{mass of ring}}$$

Reminder (re proportionality)

MATHEMATICAL ASIDE Proportionality and proportional reasoning

The concept of **proportionality** arises frequently in physics. A quantity symbolized by u is *proportional* to another quantity symbolized by v if

$$u = cv$$

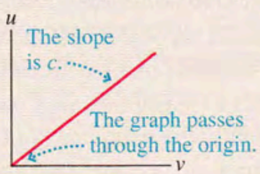
where c (which might have units) is called the **proportionality constant.** This relationship between u and v is often written

$$u \propto v$$

where the symbol $\propto$ means "is proportional to."

If v is doubled to $2v$, then u is doubled to $c(2v) = 2(cv) = 2u$. In general, if v is changed by any factor f, then u changes by the same factor. This is the essence of what we *mean* by proportionality.

A graph of u versus v is a straight line *passing through the origin* (i.e., the y-intercept is zero) with slope $= c$. Notice that proportionality is a much more specific relationship between u and v than mere linearity. The linear equation $u = cv + b$ has a straight-line graph, but it doesn't pass through the origin (unless b happens to be zero) and doubling v does not double u.



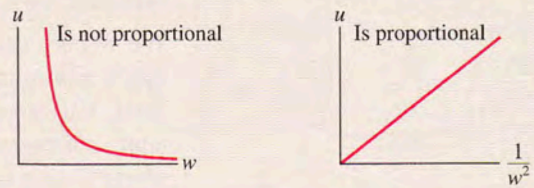
u is proportional to v.

If $u \propto v$, then $u_1 = cv_1$ and $u_2 = cv_2$. Dividing the second equation by the first, we find

$$\frac{u_2}{u_1} = \frac{v_2}{v_1}$$

By working with *ratios*, we can deduce information about u without needing to know the value of c. (This would not be true if the relationship were merely linear.) This is called **proportional reasoning.**

Proportionality is not limited to being linearly proportional. The graph on the left below shows that u is clearly not proportional to w. But a graph of u versus $1/w^2$ *is* a straight line passing through the origin, thus, in this case, u is proportional to $1/w^2$, or $u \propto 1/w^2$. We would say that "u is proportional to the inverse square of w."



u is proportional to the inverse square of w.

EXAMPLE u is proportional to the inverse square of w. By what factor does u change if w is tripled?

SOLUTION This is an opportunity for proportional reasoning; we don't need to know the proportionality constant. If u is proportional to $1/w^2$, then

$$\frac{u_2}{u_1} = \frac{1/w_2^2}{1/w_1^2} = \frac{w_1^2}{w_2^2} = \left(\frac{w_1}{w_2}\right)^2$$

Tripling w, with $w_2/w_1 = 3$, and thus $w_1/w_2 = \frac{1}{3}$, changes u to

$$u_2 = \left(\frac{w_1}{w_2}\right)^2 u_1 = \left(\frac{1}{3}\right)^2 u_1 = \frac{1}{9} u_1$$

Tripling w causes u to become $\frac{1}{9}$ of its original value.

Many *Student Workbook* and end-of-chapter homework questions will require proportional reasoning. It's an important skill to learn.

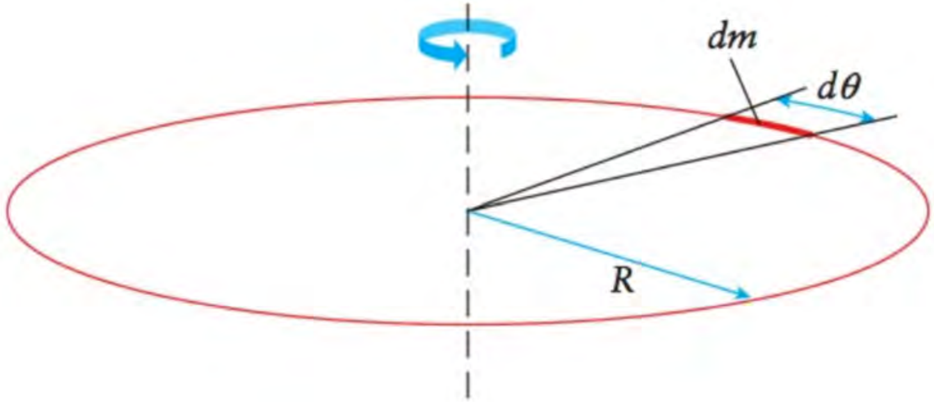
$$\frac{\text{angle of slice}}{\text{angle of ring}} = \frac{\text{mass of slice}}{\text{mass of ring}}$$

→ Because the density is uniform (i.e., constant), *proportionality* reasoning applies here!

Knight (2013)

<u>Aside</u>: Some of those other shapes....

Thin hoop rotating about center



$$I = \int r^2\, dm$$

Similar argument before re density & proportionality

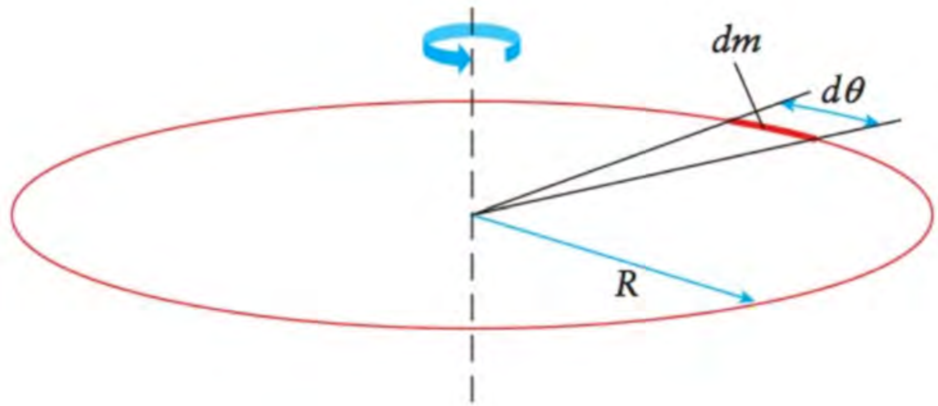
$$\frac{\text{angle of slice}}{\text{angle of ring}} = \frac{\text{mass of slice}}{\text{mass of ring}}$$

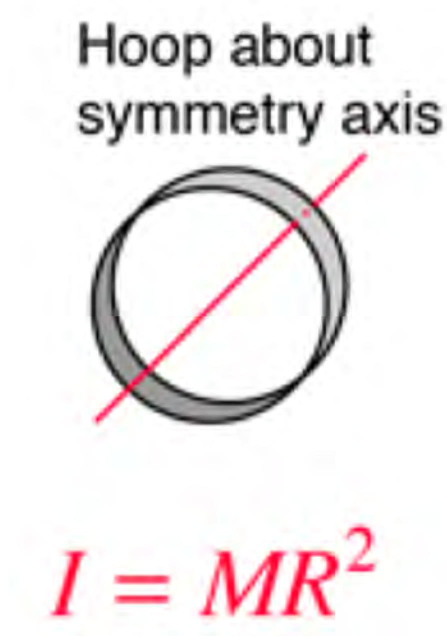
$$\frac{d\theta}{2\pi} = \frac{dm}{M} \qquad dm = \frac{M}{2\pi} d\theta \qquad I = \int r^2 \frac{M}{2\pi} d\theta = \frac{M}{2\pi} \int r^2 d\theta$$

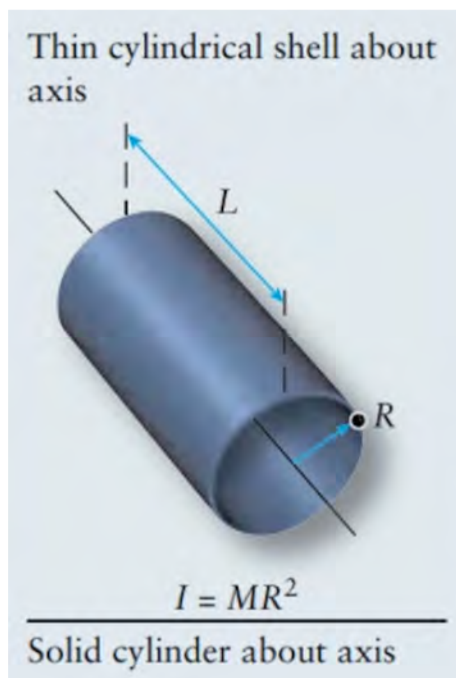
$$I = \frac{MR^2}{2\pi} \int_0^{2\pi} d\theta \qquad I = \frac{MR^2}{2\pi} \theta \Big|_0^{2\pi} = \frac{MR^2}{2\pi} (2\pi - 0) = MR^2$$

<u>Aside</u>: Some of those other shapes....

Thin hoop rotating about center



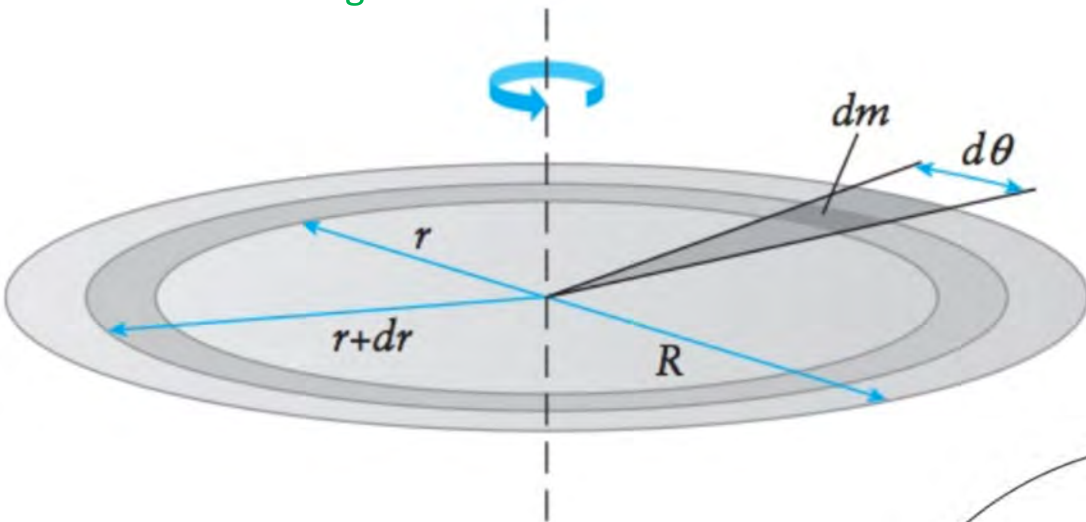


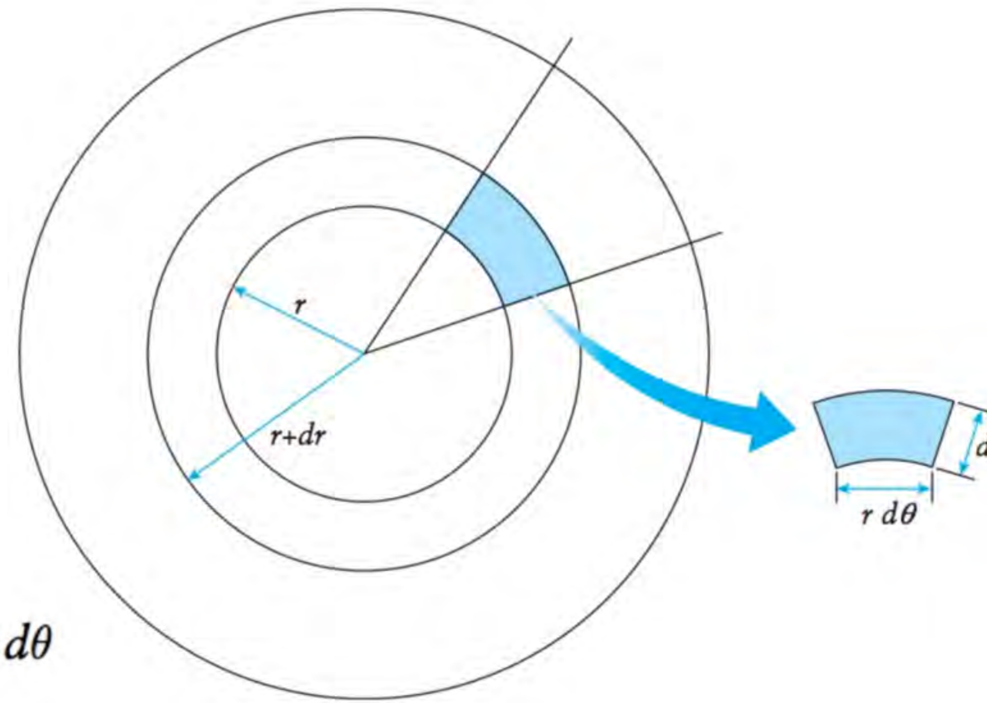


→ Remarkably L does not factor in here!

Aside: Some of those other shapes....

Thin disk rotating about center



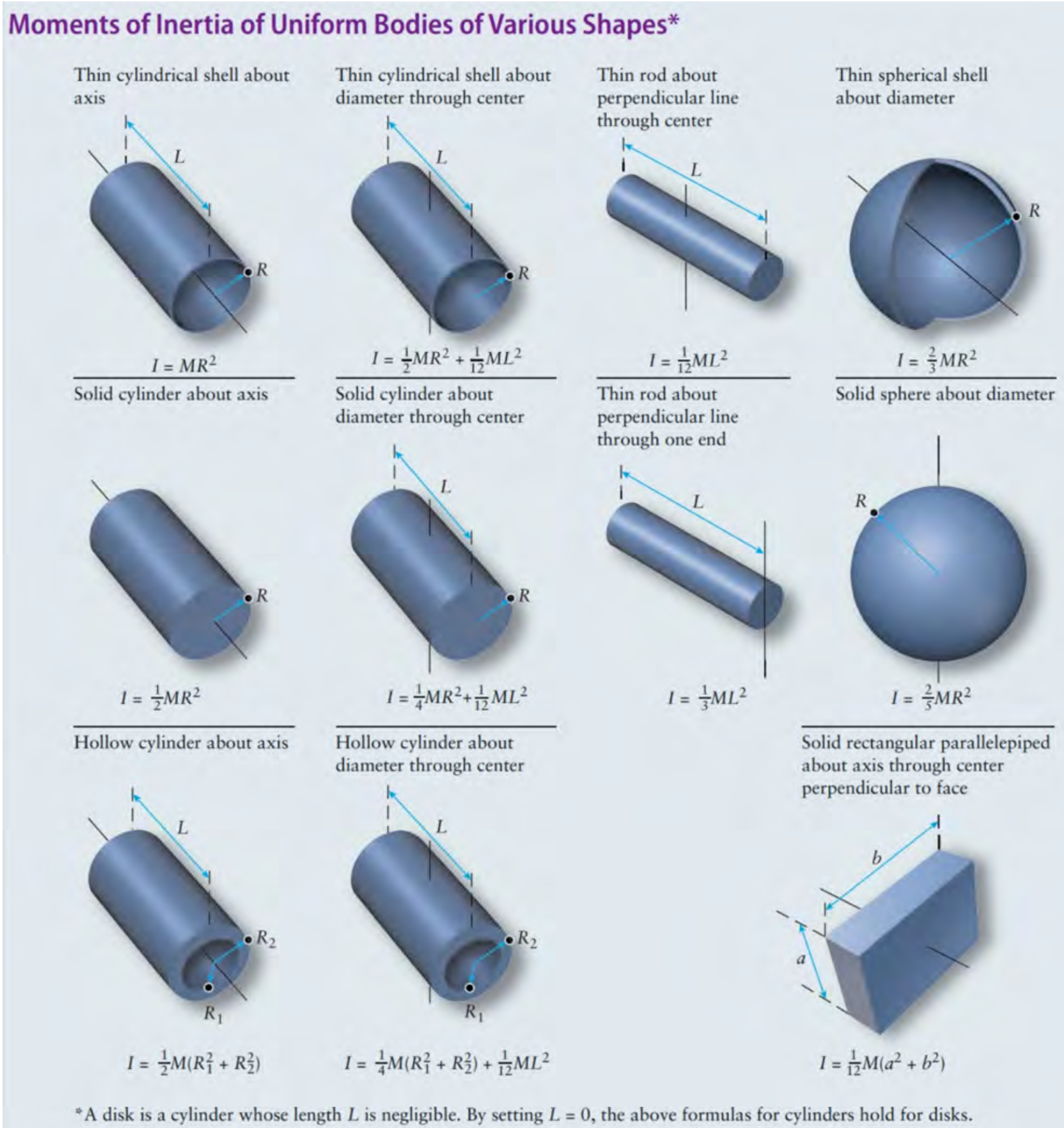


Similar argument before re density & proportionality

Double integrals! (beyond our scope)

$$I = \int \int r^2 \frac{M}{\pi R^2} r\, dr\, d\theta = \frac{M}{\pi R^2} \int \int r^3 dr\, d\theta$$

Kesten & Tauck



Kesten & Tauck

- Tells you how the moment of inertia would change for one of those regular objects it we chose to rotate the object about some other axis.

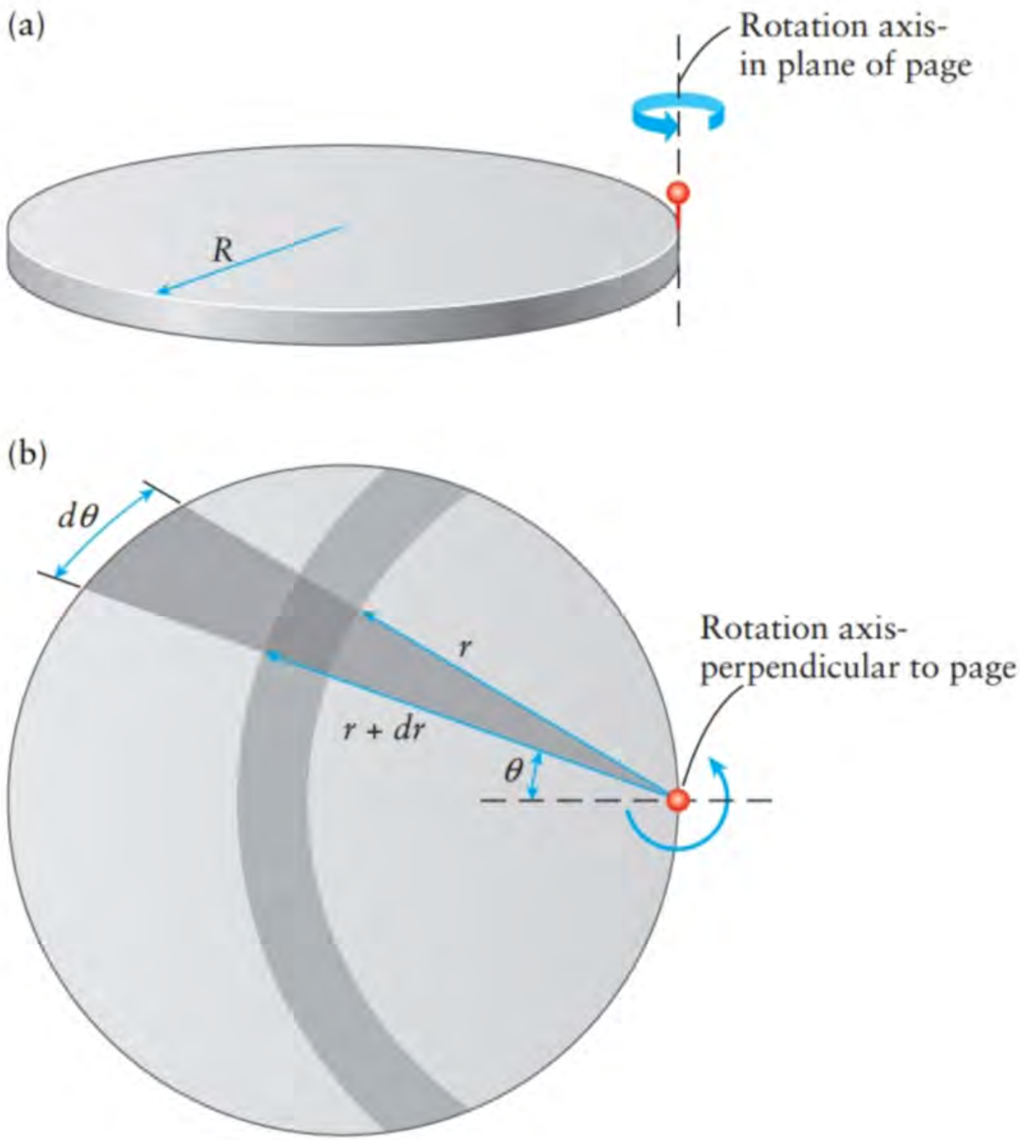
$$I = I_{CM} + Mh^2$$

where I_{CM} is the moment of inertia through the centre of mass as indicated in some table.

M is the mass of the object.

h is the distance of from the centre of mass to the new axis of rotation.

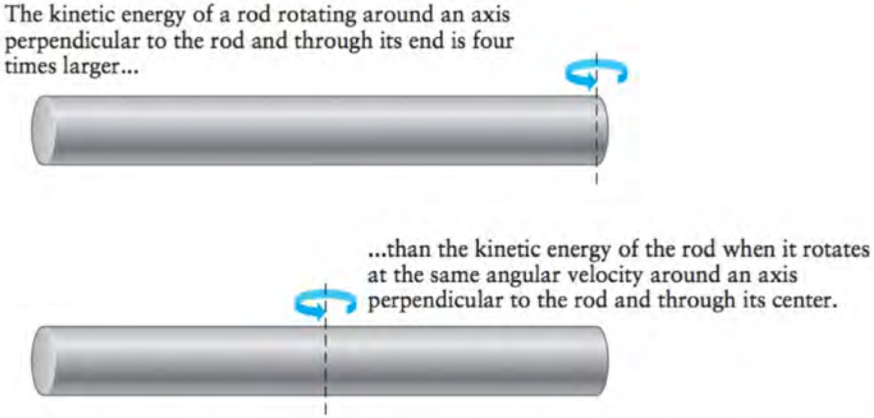
Parallel-Axis Theorem



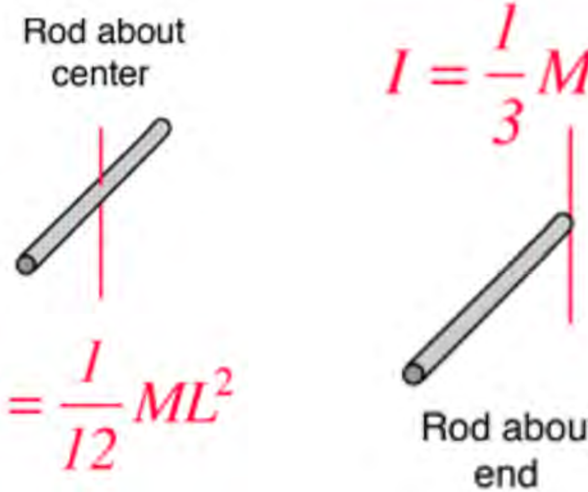
$$I = I_{CM} + Mh^2$$

Kesten & Tauck

<u>Parallel-Axis Theorem</u>: (quasi-Proof)



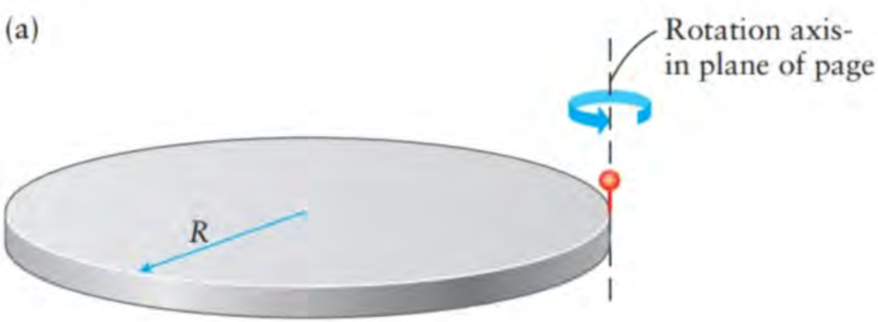
The more mass there is farther from the rotation axis, the larger the moment of inertia, and the larger the rotational kinetic energy for any given angular velocity.



$$I = \frac{1}{12}ML^2 \qquad I = \frac{1}{3}ML^2$$

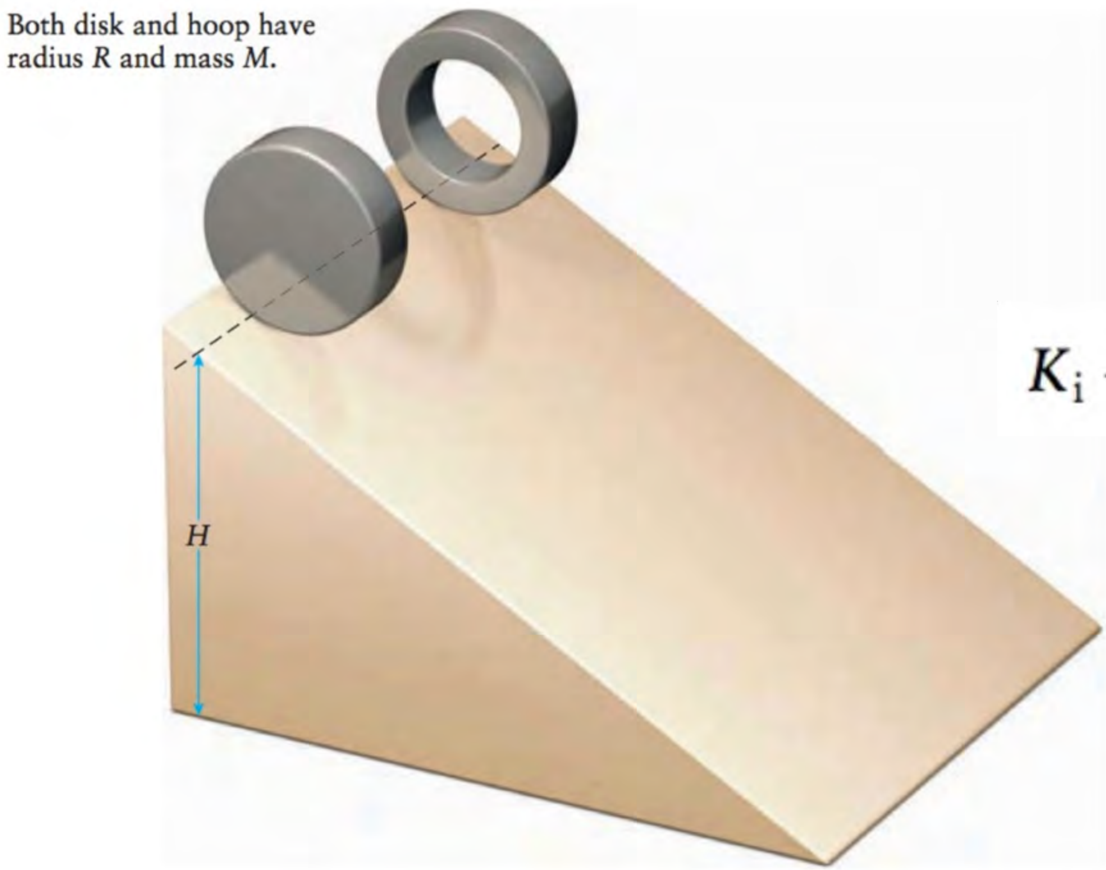
$$I = I_{CM} + Mh^2$$

$$\begin{aligned} I &= \frac{ML^2}{12} + M\left(\frac{L}{2}\right)^2 \\ &= \frac{ML^2}{12} + \frac{ML^2}{4} \\ &= \frac{ML^2}{12} + \frac{3ML^2}{12} \\ &= \frac{4ML^2}{12} \\ &= \frac{ML^2}{3} \end{aligned}$$



Kesten & Tauck

Conservation of Energy (Revisited)



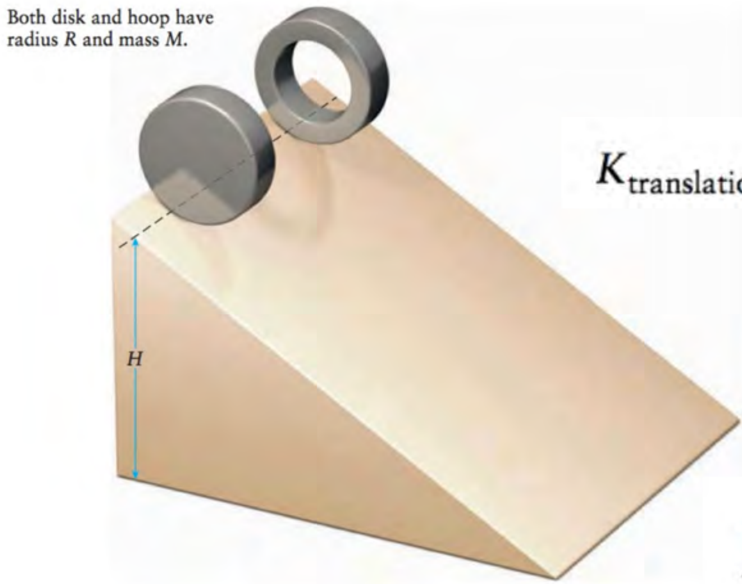
$$K_i + U_i = K_f + U_f + |W_{nc}|$$

Now we factor rotation into the kinetic energy consideration:

$$K_{translational,i} + K_{rotational,i} + U_i = K_{translational,f} + K_{rotational,f} + U_f + |W_{nc}|$$

Conservation of Energy (Revisited)

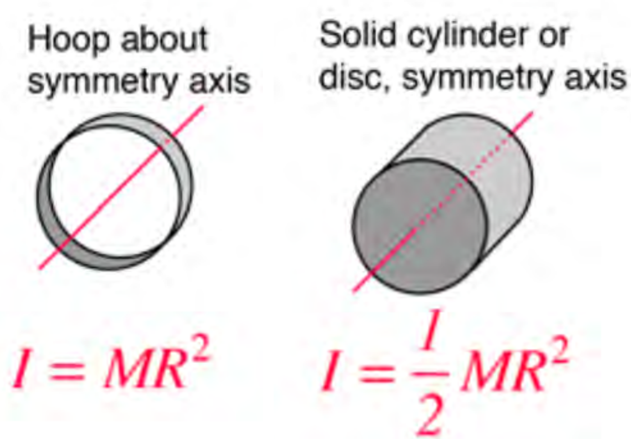
- Ignore losses
- Consider one object at a time
- Assume it is initially at rest



$$K_{\text{translational, i}} + K_{\text{rotational, i}} + U_i = K_{\text{translational, f}} + K_{\text{rotational, f}} + U_f$$

$$Mgh_i = \frac{1}{2}Mv_f^2 + \frac{1}{2}I\omega_f^2 + Mgh_f$$

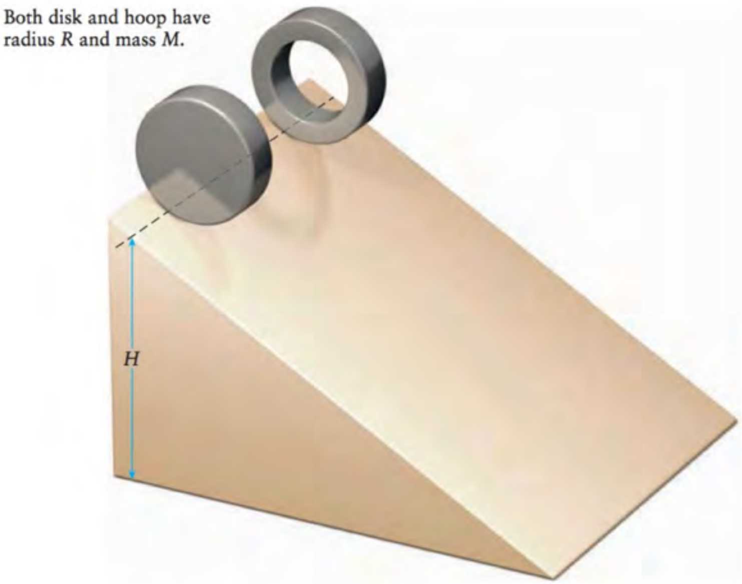
$$\frac{1}{2}Mv_f^2 = Mgh_i - Mgh_f - \frac{1}{2}I\omega_f^2 = MgH - \frac{1}{2}I\omega_f^2$$



$$v = \left(\frac{2\pi}{\Delta t}\right)R = \omega R$$

Conservation of Energy (Revisited)

When the smoke clears....



$$v_{\text{disk, f}} = \sqrt{\frac{4}{3}gH}$$

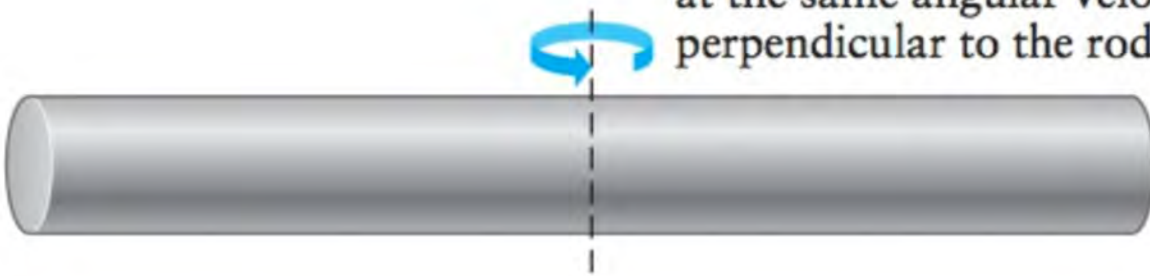
$$v_{\text{hoop, f}} = \sqrt{gH}$$

The "disk" wins the race. Perhaps not very intuitive at first....

... until you keep in mind a key principle at play here:
Energy can be stored in a variety of ways

The kinetic energy of a rod rotating around an axis perpendicular to the rod and through its end is four times larger...

...than the kinetic energy of the rod when it rotates at the same angular velocity around an axis perpendicular to the rod and through its center.



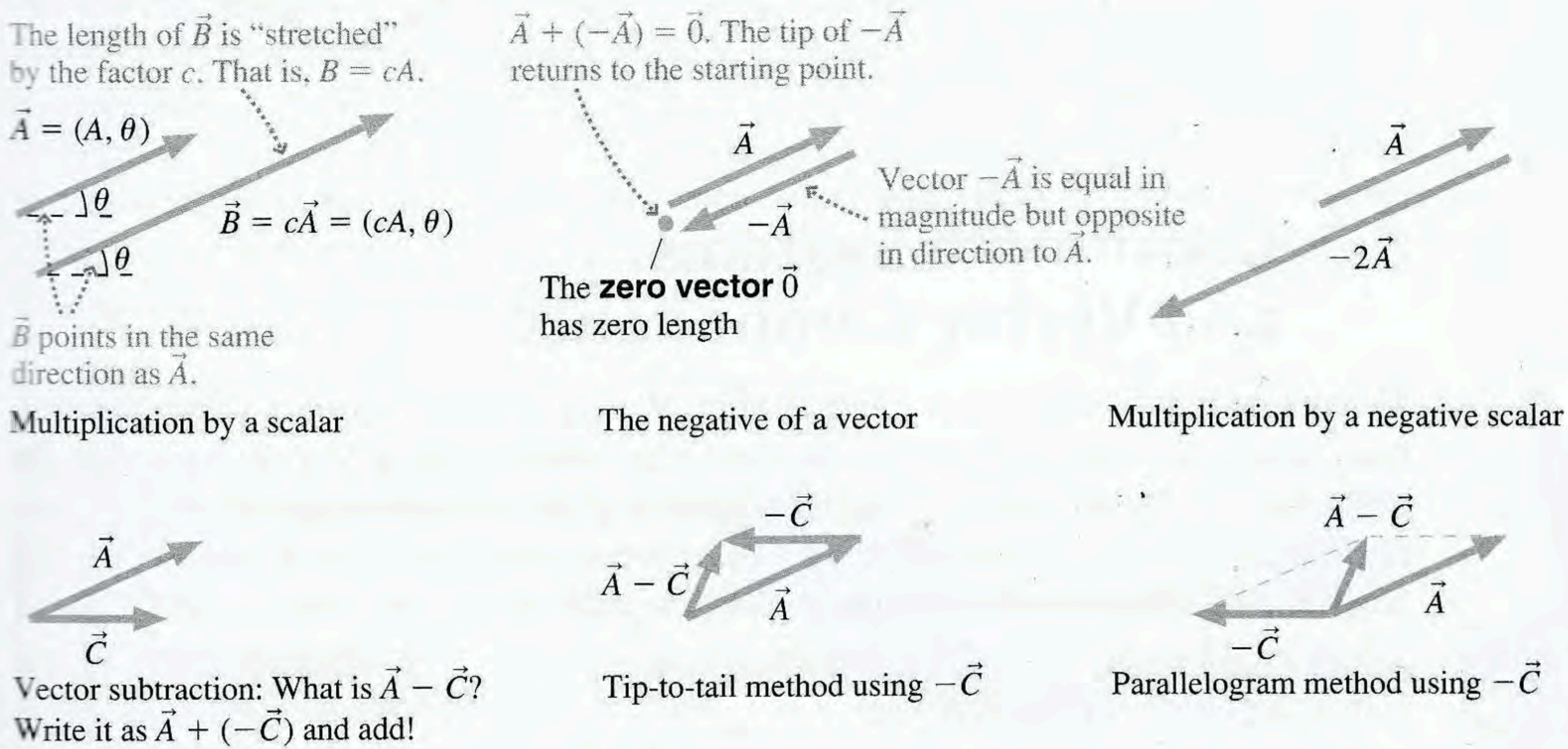
The more mass there is farther from the rotation axis, the larger the moment of inertia, and the larger the rotational kinetic energy for any given angular velocity.

Kesten & Tauck

<u>Review</u>: Vector Algebra

Vector algebra follows familiar rules (cont)

FIGURE 3.7 Working with vectors.



<u>Note</u>: Multiplication of two vectors is a bit trickier (we'll get there later in the semester)

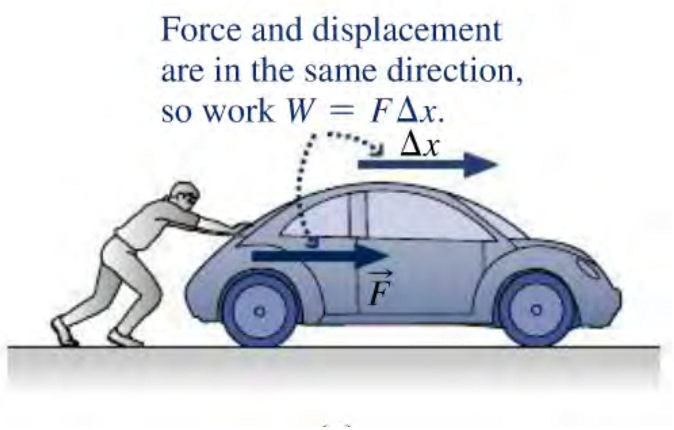
Knight

Review: Vector Algebra

The "dot product" (re work)

$$W = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r}$$

$$\vec{A} \cdot \vec{B} = AB\cos\theta$$



Wolfson

→ Another means to multiply vectors (the "*cross product*") now arises as we head into the notion of torque...

$$\vec{\tau} = \vec{r} \times \vec{F} = |\vec{r}|\,|\vec{F}| sin\phi$$

where $\vec{r}$ is the distance from the axis of rotation

$\vec{F}$ is the applied force

ϕ is the angle between $\vec{r}$ and $\vec{F}$

Torque

- But how does one achieve a particular angular speed? What cause angular acceleration?

- For translational motion, it was a force that caused changes in velocity.

- For rotational motion, it is the torque, $\vec{\tau}$, (tau) that causes changes in angular speed. Torque can be thought of as the angular force.