Reverse-Engineering the Copernican Revolution: Exploring How Inverse Problems Lead to Models

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Abstract: Centuries of careful observation of the nighttime skies prior to the 16th century could not get around one simple fact: Our limited observational perspective and desire to develop a physical theory of the heavens essentially amounted to an inverse problem. The Copernican Revolution, the shift from a geocentric to heliocentric model of the solar system, represented a watershed in scientific theory as it cleared away the overly complicated Ptolemaic approach and brought forth a more coherent and encompassing framework. Here, we were motivated by the ubiquity of inverse problems in modern science and aim to gain insight into how successful theoretical frameworks are initially developed. Towards this end, we developed a simple model consisting of a point-projection of two nested phase oscillators. Despite its simplicity, the behavior of the model exhibited remarkable complexity. One approach taken was to consider what other types of 'models' would produce similar behavior. That is, though we already had the 'answer', we (re-)created an inverse problem to examine the range of possible models that could have arisen. Using the 'paradigm shift' led by Copernicus as a lens, we attempted to tie our analysis back to understanding what strategies exist to optimize the decisions leading towards the development of successful theoretical frameworks.

Background

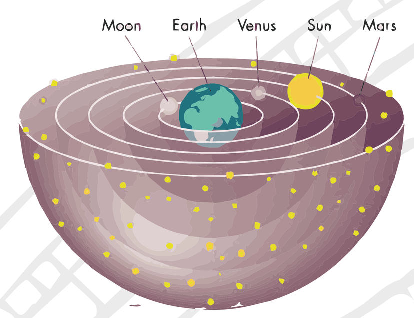


FIGURE 1 - Geocentric model of the heavens [redorbit.com].

Copernican Revolution - Mankind has long pondered the heavens and strived to come up with a conceptual framework (i.e., a model) for understanding such. Up until the 17th century, the dominant view was a geocentric model where the Earth sat in a collection of nested spheres (Fig.1). Complexities, such as the retrograde motion of Mars (Fig.2) could be explained in the Ptolemaic system by means of epicycles (Fig.3). Such an idea could be extended (Fig.4) to describe even more complex observations. As the models got more complicated, it became harder to assert their validity [Kuhn]. Pioneered in part by Copernicus, a 'shift' took place towards a heliocentric model (Fig.5,15). A key advantage with this major change in physical assumptions was a much simpler conceptual framework more capable of explaining the wide range of observational data.

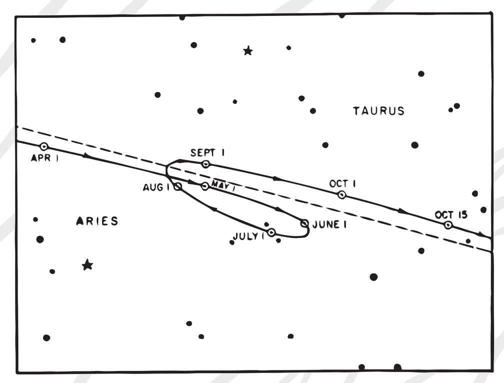


FIGURE 2 - Retrograde motion of Mars [Kuhn, 1992].

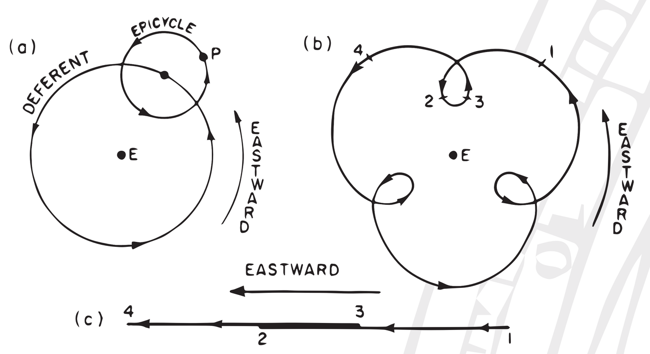


FIGURE 3 - Overview of an epicycle-deferent model. Panel B shows the looped motion generated in the plane of the ecliptic, while panel C shows a portion (1-2-3-4) of the motion in B as it is seen by an observer on the central earth, E [Kuhn, 1992].

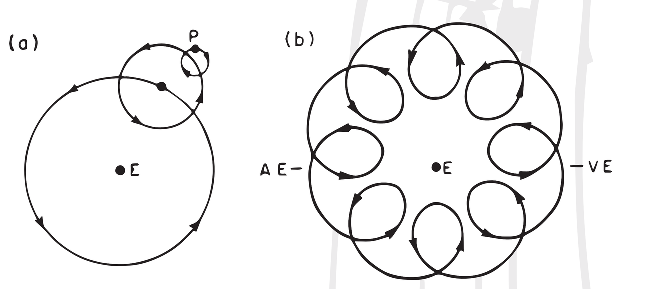


FIGURE 4 - Epicycle on an epicycle on- a deferent. Successively complicated behavior can be obtained by such a theoretical framework [Kuhn, 1992].

Inverse Problems - The basic issue was that the development of an astronomical model essentially amounted to an ill-posed question: Given our limited observational point of view, can we come up with a all-encompassing model of the universe? The challenge with 'inverse problems', where you know the solution but not necessarily the question being asked, is that there is often (many) more than one suitable answer. Consider that while the Ptolemaic model provided a seemingly reasonable answer to the problem of retrograde motion, the basic underlying assumption proved to be fundamentally wrong. Given that widely accepted models generally carry a large degree of momentum in the scientific community, it typically takes a good deal of time/energy/resources to make corrections (Kuhn's 'paradigm shift').

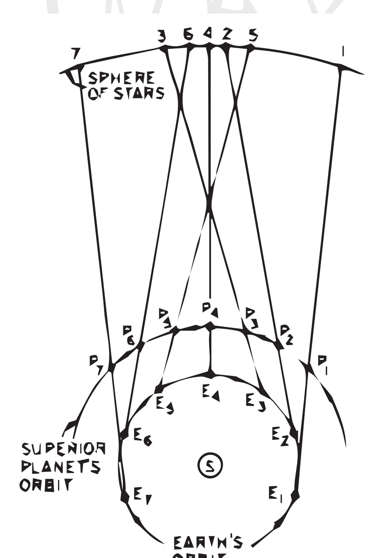


FIGURE 5 - Refined model: Copernicus' heliocentric framework for retrograde motion - Here, the earth (E) revolves around the sun (S), as does a superior planent (P) (e.g., Mars). The apparent progression of the planet upon the stellar sphere exhibits its a brief retrograssion (from 3 to 5) [Kuhn, 1992].

Questions - Motivated by the fact that many areas of modern science deal with inverse problems, our goal here was to create a simple model and examine the pitfalls/challenges associated with having to reverse-engineer it from knowledge of the model's output alone. For example, how complex can the observed behavior of a simple system become? What sort of conceptual frameworks might give rise to similar behaviors? How fundamentally different are these types of models from one another?

Methods

➤ Drawing inspiration from the London Eye (Fig.6), we developed a model consisting of two nested phase oscillators (Fig.7), each with constant angular velocities. The model 'output' $I(t)$ was derived from the digital projection of a line onto a point (Figs.7,8), akin to a '0-D Radon transform' (i.e., an integral transform onto a straight line, convolved with a delta function).

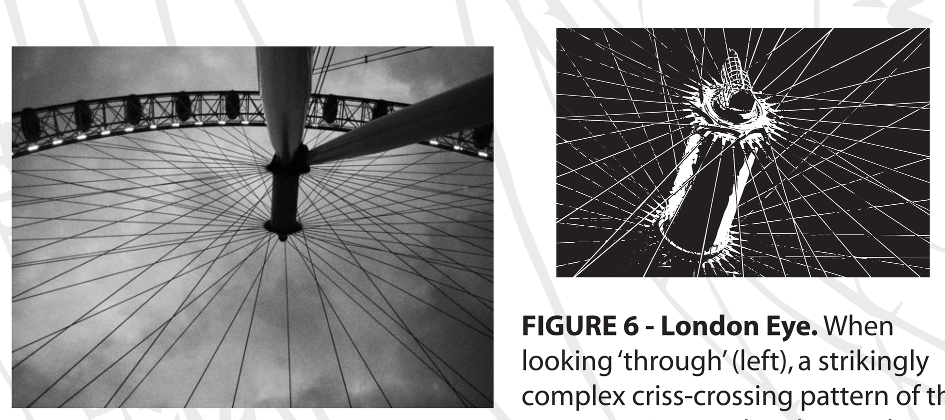


FIGURE 6 - London Eye. When looking 'through' (left), a strikingly complex criss-crossing pattern of the support wires can be observed, despite a straight-forward design (top).

➤ Model simulations were done using Matlab. One strategy to examine the model's output was to assess the presence of periodicity. This was done via the development of an auto-correlation function (ACF), which is the comparison of a segment of itself at one point in time to itself at all other points in time. The normalized ACF here was defined as:

$$\mathrm{ACF}(t) = \frac{\sum_{\tau} I(\tau)I(\tau - t)}{\Delta} \quad \text{where} \quad \Delta \equiv \sum_{\tau} I^2(\tau) \quad \text{(i.e., at } t=0\text{)}$$

When periodicities were present, the ACF exhibited global maximums (ACF=1) that could be used to infer repeatability.

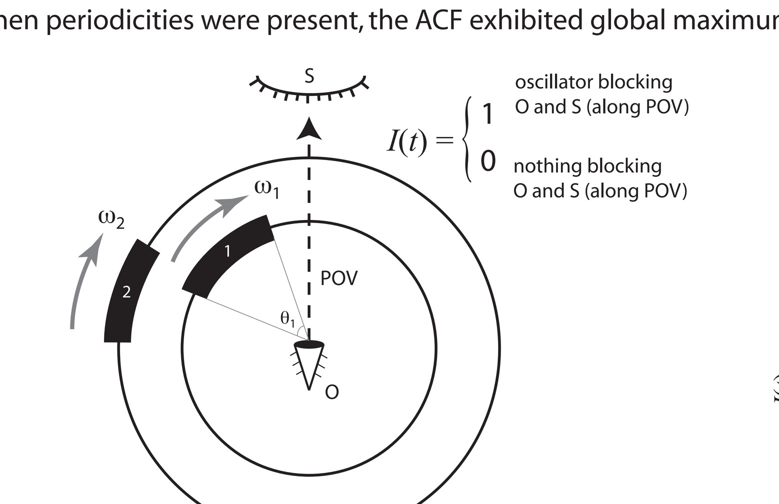


FIGURE 7 - Schematic of model . An observer (O) is positioned at the center and looks 'out' with point-of-view (POV) towards a source (S). Around the observer rotate two objects: each spans some radial distance and has a unique (constant) radial velocity.

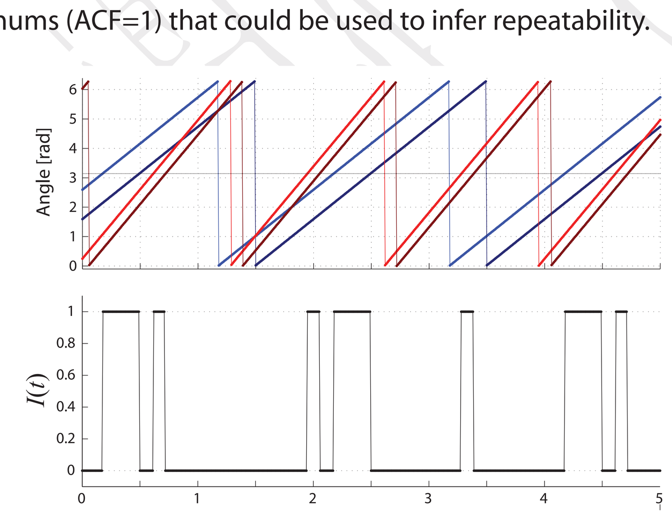


FIGURE 8 - Schematic of model output. Oscillators properties: ω_1 =0.75 cycles/s, θ_1 =0.5 rad, ω_2 =0.5, θ_2 =1.

Results

➤ The observed pattern of $I(t)$ exhibited complex behavior. The effect upon the periodicity of $I(t)$ for different conditions are summarized as follows:

• Single oscillator	rational ω	→ periodic signal	
• Two oscillators	rational ω_1, ω_2	→ periodic signal	(Fig. 9)
• Single oscillator	irrational ω	→ periodic signal	
• Two oscillators	irrational ω_1, ω_2	→ aperiodic signal	(Fig. 10)

➤ The nature of the POV only allows observation of the compound (i.e., superpostion of two oscillators) and not the individual periods. We surmise that the compound period (T) may be computed by $T = \mathrm{LCM}[T_1, T_2]$, where $T = 2\pi/\omega$ and LCM stands for the Least-Common-Multiple. Note that T does not exist if T1 and T2 are irrational (i.e, an aperiodic signal is expected to arise, consistent with Fig.10). A function defined by an LCM is expected to vary with its arguments in a complex fashion (Fig.11, black curve). Periodicities extracted from simulations confirm this prediction (Fig.11, red curve). Note that this curve is subject to sudden changes which would be a challenge to characterize had the input periods been unknown.

➤ The observer only creates a digital signal. Thus, the analog position of the object would have little impact on the output. However, if small mechanical disturbances are added to the system such that the position of the object spontaneously changes by nominal amounts, it should be observable that the system is sensitive to noise near the POV and not at all far from the POV. This is because near the POV the slight shift is able to move the object either into or out of the POV thus completely changing the output.

➤ As shown in Fig.12, an increasing degree of variation in the initial starting velocity of one of the oscillators causes an increasing effect upon $I(t)$ (e.g., change in compound period, loss of periodicity). Thus by and large, the system does exhibit a degree of sensitivity to initial conditions.

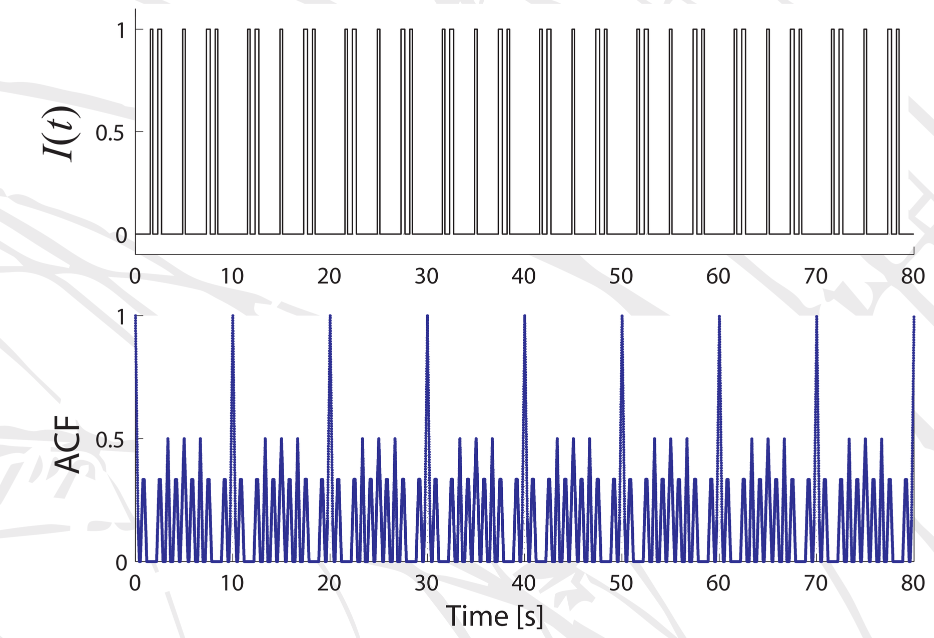


FIGURE 9 - Rational frequencies. Oscillators properties: T_1 = 5 s, θ_1 = 0.5 rad, T_2 = 0.5, and θ_2 = 0.5. Note that $T = 2\pi/\omega$.

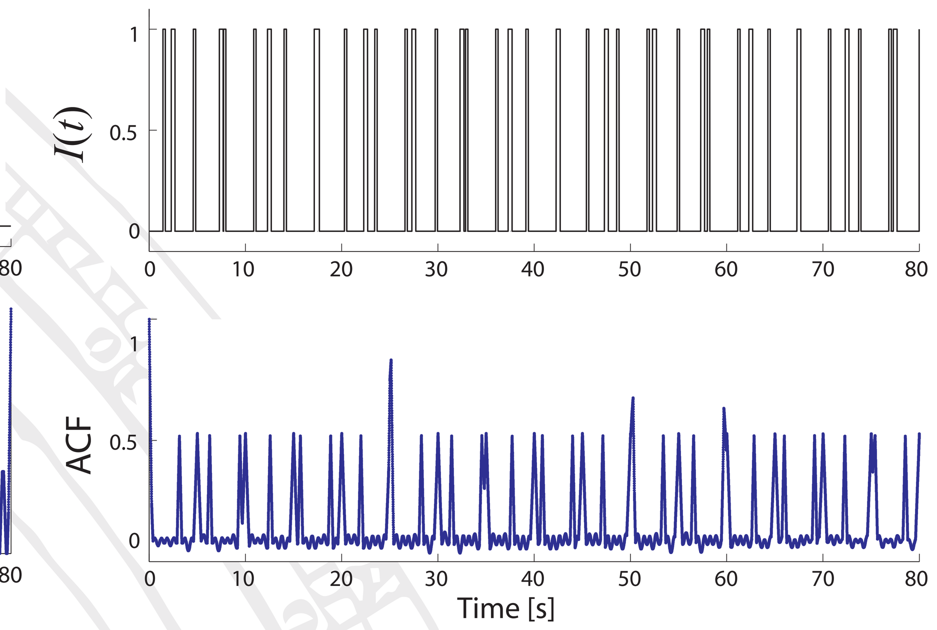


FIGURE 10 - Irrational frequencies. Oscillators properties: T_1 = 5 s, θ_1 = 0.5 rad, T_2 = π, and θ_2 = 0.5.

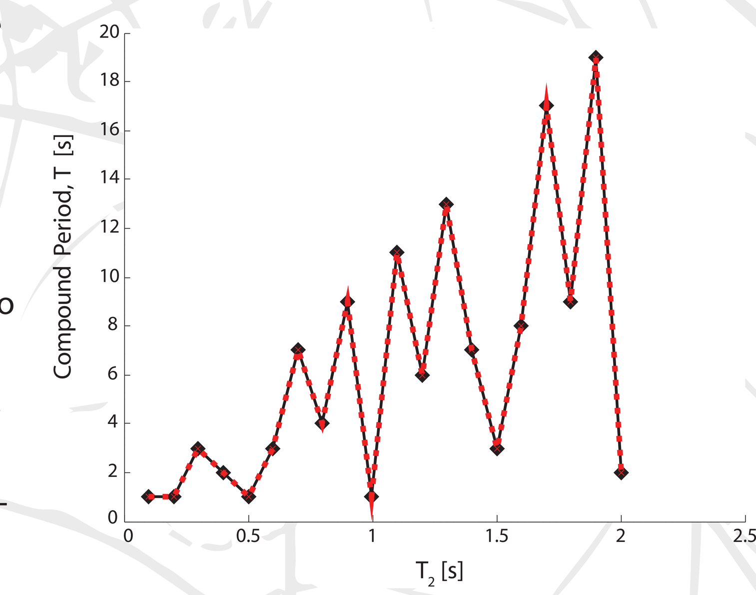
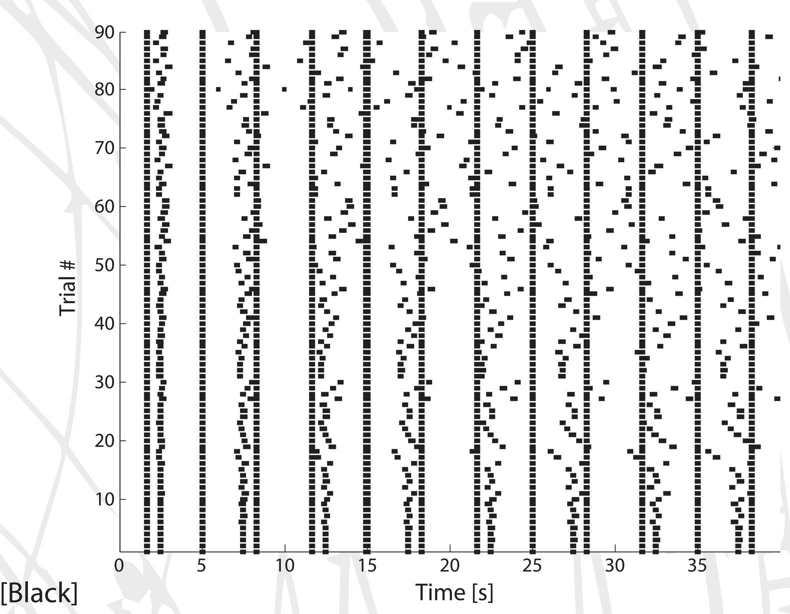


FIGURE 11 - Comparison of model periodicity and LCM. [Black] Change in the Least-Common-Multiple (LCM) with respect to T_2 while keeping T_1 fixed at 1 s. [Red dashed] Periodicity extracted from model via ACF for similar manipulations of the periods.



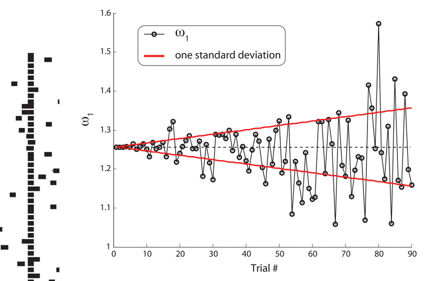


FIGURE 12 - Sensitivity to (noisy) variation in initial conditions. Raster plot showing 90 trials (left), where a variation (sampled from a normal distribution) in T_1 was gradually increased from 0 to 10% of the base value (top) from bottom to top.

Oscillators (base) properties: T_1 = 5 s, θ_1 = 0.5 rad, T_2 = 10/3, and θ_2 = 0.7. T_2 was held constant. Note that the noisy oscillator can be compared to a stable one by virtue of the regular vertical lines beside the irregular ones.

Discussion

Simple models can exhibit complex behavior - Despite the simplicity of our model (Fig.7), the output can be strikingly complex (e.g., Figs.9-12). For example, the combination of two periodic components can give rise to something aperiodic. A chief consideration here is that we are assessing the system in a very simplistic way: the observational point of view is a single point and effectively ignores most of the information about the oscillators (thereby creating apparent complexity). However, such a facet bears consideration in that much of observational science typically takes a limited point of view of that which is being observed. Furthermore, it is not hard to envision how a slightly more complicated model (e.g., addition of third oscillator) could give rise to further complexities in the output. Consider again the London Eye and a projection through the support wires as the wheel rotates.

Linearity/Nonlinearity - Nonlinearity can give rise to highly complicated dynamics, even for 'simple' systems (e.g., Lorenz attractor, cellular automata). One question thus worth exploring is to what extent our model (as described in Fig.7) is linear or not. The oscillators themselves are linear, as would be any transforms that completely map the system (e.g., a 1-D Radon transform that projected the motion onto a straight line). However, given the way that $I(t)$ is defined (i.e., a point projection), we surmise that the system is not strictly linear. To what extent this aspect may in fact be responsible for the observed 'complexity' remains to be determined.

Other potential models? - We conjecture here other (broadly classed) types of models that could exhibit similar dynamics to those of our nested oscillator model. These include:

- Passing of blades from a system of overlapped wind turbines through a point
- Digital logic circuit with an AC driven gate
- Flow of traffic due to pedestrians in a crosswalk with no traffic lights
- Crossing of a double pendulum back over itself (Fig 13])
- Spiking signals from a firing neuron (whether alone or embedded in a network)

Though potentially governed by different mechanics, these systems can exhibit both periodic and aperiodic regimes in their 'digital' output

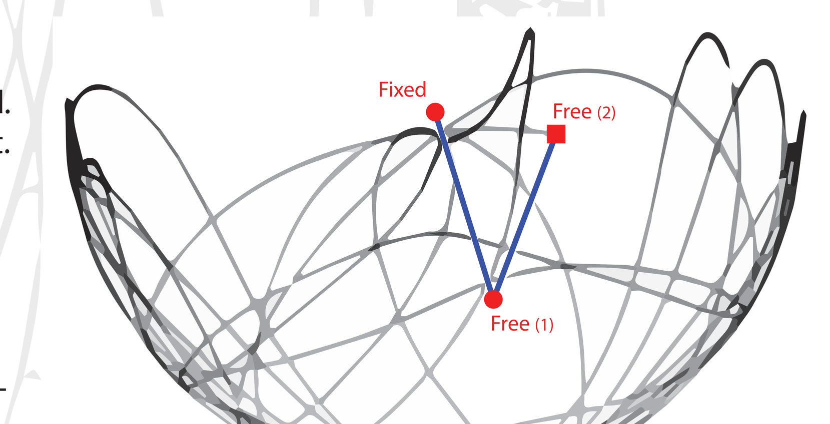
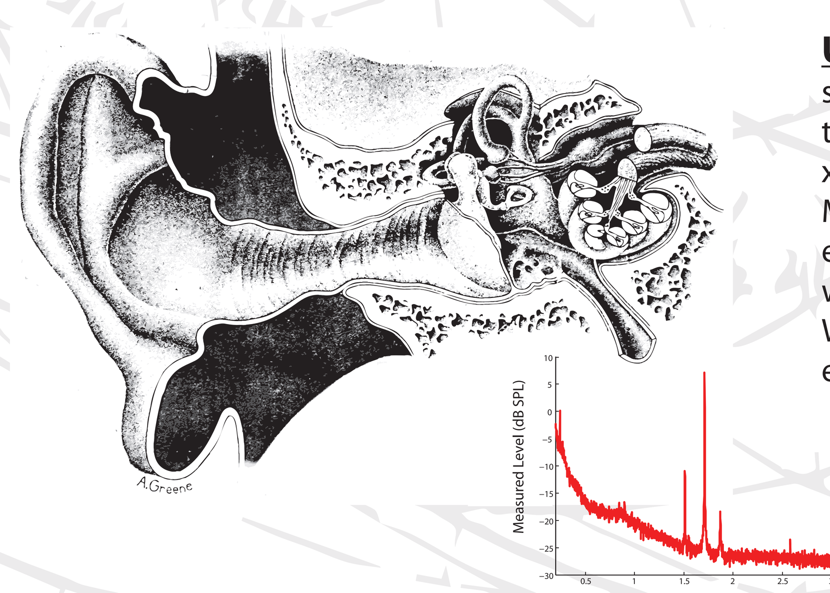


FIGURE 13 - Double-pendulum. Snapshot of time evolution of the free end (2). Shading corresponds to velocity (darker = slower, lighter = faster). [Adapted from Paul Nathan]



Ubiquity of inverse problems in science - Aside from astronomy, many classical physics experiments essentially dealt with inverse problems. For example, the bombardment of gold foil by alpha particles to determine the atomic structure by E. Rutherford is a classic example (aka the Geiger–Marsden experiment). Similarly, modern x-ray crystallography techniques used to determine atomic/molecular structures share a similar nature (Fig.14). Many other examples abound, such as using the emission of sound (otoacoustic emissions, OAEs) from healthy ears as a means to probe cochlear mechanics (Fig.14 inset). Many of these problems have a common foundation: we are looking at a single measure from systems whose underlying properties are substantially more labyrinthine. When using such data to construct models, it may not be immediately obvious if we have sufficiently observed essential features that allow for optimized decisions upon chosen assumptions.

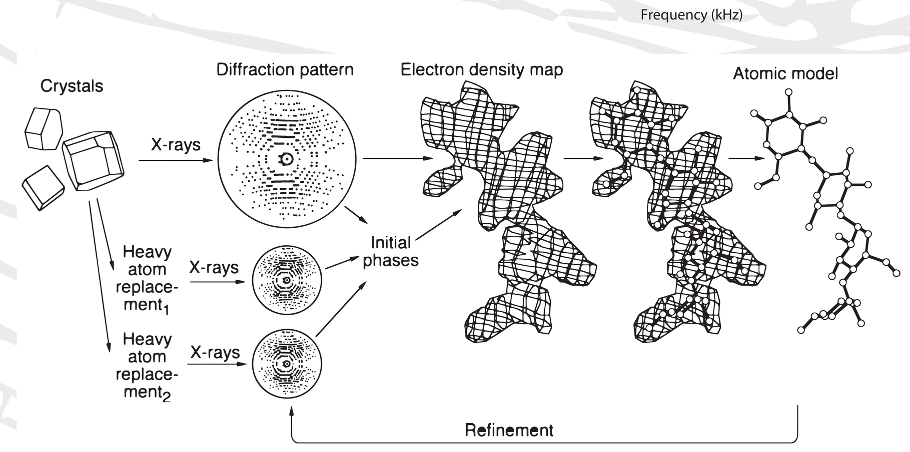


FIGURE 14 - Examples of other types of inverse problems. Top plot shows an overview of the peripheral auditory system and an example of a spontaneous otoacoustic emission (red curve). Bottom schematic shows an overview of x-ray crystallography, a technique commonly used for identifying atomic/molecular structure (e.g., proteins) [Nolting].

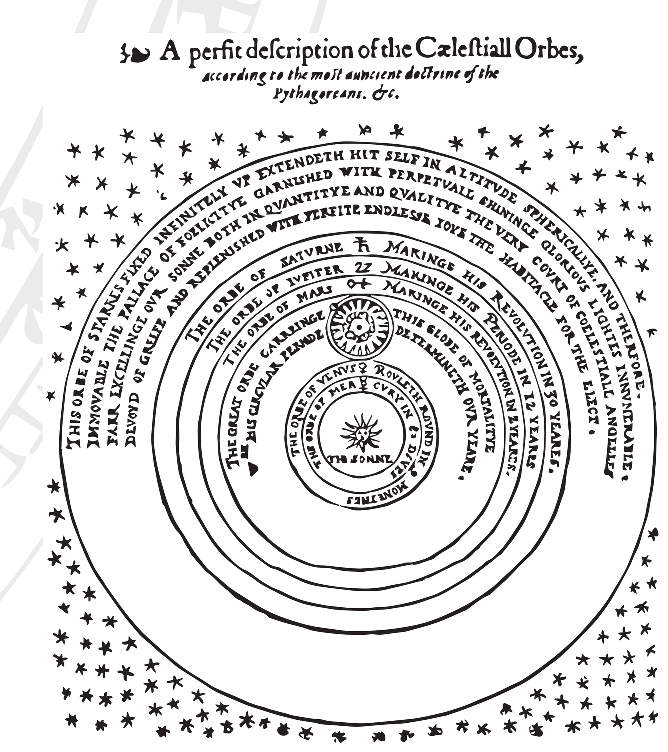


FIGURE 15 - The infinite Copernican universe of Thornas Digges (*Perfit Description of the Caelestiall Orbes*,1576) [Kuhn, 1992].

Conclusions

Testability - Based upon $I(t)$ alone, development of a model to describe such is a fundamentally ill-posed question, as there are likely an infinite number of possibilities. To get something meaningful, any model developed has to be able to make testable predictions that can be used to verify/nullify a specific hypothesis. The 'problem solving procedure' of Polya is also valuable to consider with regard to model building [Polya, 2004].

Paradigm Shifts - Connecting back to the 'Copernican Revolution': History has taught us that it can take long periods of time for changes in thought once a reasonable model has taken hold (i.e., Kuhn's 'paradigm shift'). Thus it is of critical importance to remember that we must continually critically assess our underlying assumptions and their validity

Occam's Razor - In striving for the most efficient comprehensive model, it is worthwhile to keep in mind the *lex parsimoniae* of William of Ockham:

One should proceed to simpler theories until simplicity can be traded for greater explanatory power.

While more of a sound guiding principle rather than a fundamental rule, perhaps Albert Einstein put it rather more succinctly/effectively: "Keep it as simple as possible, but not simpler." Determining the simplest effective hypothesis is one of the greatest challenges scientists face.

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