Amenable actions of the infinite permutation group — Lecture IV

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The main question left open is the following which, of course, does not assume any extra set theory beyond Choice.

Question

Is there an amenable subgroup of $\mathbb{S}(\omega)$ whose natural action on ω has a unique mean?

But there are a number of intermediate questions that may be of interest as well.

Question

Is there a model where $\mathfrak{p} \neq \mathfrak{u}$ yet the Key Hypothesis still holds? Recall that the **Key Hypothesis** *is the following: There is a generating set $\{G_\xi\}_{\xi\in\kappa}$ for an ultrafilter on ω such that there exist infinite $A_\xi \subseteq \omega$ satisfying:*

- $A_\xi \subseteq^* G_\eta$ *for each* $\eta \leq \xi$
- $A_\xi \cap A_\eta$ *is finite if* $\xi \neq \eta$.

Is there such a model where no ultrafilter is generated by a tower?

Question

Is there a model where the Key Hypothesis fails yet there is still an amenable subgroup of $\mathbb{S}(\omega)$ whose natural action on ω has a unique mean?

Foreman has shown that assuming Martin's Axiom every amenable group of size less than $\mathfrak{c}$ has $2^{\mathfrak{c}}$ invariant means. A weaker version of the main question is:

Question

Is there an amenable subgroup of $\mathbb{S}(\omega)$ whose natural action on ω has less than $2^{\mathfrak{c}}$ invariant means?

Until this point only discrete groups have been discussed. Pestov asked whether an amenable topological group acting on $\mathbb{B}$ can have a unique invariant mean. A modification of Foreman's construction provides a positive answer.

Let $\{\xi_n\}_{n=0}^{\infty}$ enumerate all finite one-to-one functions from $\mathbb{N}$ to $\mathbb{N}$. For any partial partial involution σ let $\overline{\sigma}$ denote the permutation that agrees with σ on its domain and is the identity elsewhere, Construct by induction on n sets $\{A_n^j\}_{j=0}^{\infty}$ and involutions $\sigma_n^j : A_n^j \to A_n^{j+1}$ such that

1. $\mathbb{N} \setminus \bigcup_{i=0}^{n} \bigcup_{j=0}^{J} A_i^j$ is infinite for each J
2. $A_n^j \cap A_n^k = \emptyset$ of $j \neq k$.
3. If $A_n = \bigcup_{j=1}^{\infty} A_n^j$ then $A_n \cap A_m^j$ is finite for $m < n$ and all j.
4. If there are $\{B^j\}_{j=0}^{\infty}$ such that setting $A_n^j = B^j$ yields that (1), (2) and (3) hold and if for some j there is an involution $\theta : B^j \to B^{j+1}$ such that $\xi_n \subseteq \overline{\theta}$ then $\xi_n \subseteq \overline{\sigma}_n^j$.

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Let $\mathbb{A}$ be the group generated by $\{\overline{\sigma}_n^j\}_{n,j\in\omega}$.
Extend the ideal generated by $\{A_n^j\}_{n,j\in\omega}$ to a maximal ideal $\mathcal{J}$ and for each $X \in \mathcal{J}$ choose A_X such that $A_X \cap X = \emptyset$ and $A_X \cap A_n^j$ is finite for all n and j. This is possible because $\left\{A_n^j \setminus X \mid n, j \in \omega\right\}$ generates a proper ideal on the complement of X. Let $\{A_X^j\}_{j=1}^{\infty}$ partition A_X into infinite sets and choose $\theta_X^j : A_X^j \to A_X^{j+1}$ to be involutions. Let $\mathbb{G}$ be the group generated by $\mathbb{A} \cup \{\overline{\theta}_X^j\}_{X\in\mathcal{J},j\in\omega}$.
Let $\mathbb{G}$ inherit the topology of pointwise convergence from the full symmetric group.

- $\mathbb{A}$ is dense in $\mathbb{G}$.
- $\mathbb{A}$ is locally finite and hence amenable. Hence $\mathbb{G}$ is also amenable.
- The natural action of $\mathbb{G}$ on $\mathbb{N}$ is uniquely amenable.

Given that all the arguments to this point have relied on group being locally finite in order to claim it is amenable on can ask whether the action of a non-locally finite amenable subgroup of $\mathbb{S}(\omega)$ can have a unique mean. To eliminate trivialities, on can ask the following:

Question

Is there an amenable subgroup $G \subseteq \mathbb{S}(\omega)$ whose action has a unique mean and such that every non-identity element of G has infinite order? Consistently?

Define an indexed set $\{\sigma_i\}_{i\in n}$ of permutations of ω to be *growing* by induction on n. Let $F(\sigma)$ denote the fixed points of a permutation σ. If $n = 0$ then $\{\sigma_i\}_{i\in 0}$ is growing. If $n > 0$ then $\{\sigma_i\}_{i\in n}$ is growing if

- $\{\sigma_i\}_{i\in n-1}$ is growing
- for each $k \in \omega \setminus F(\sigma_{n-1})$ the orbit of k under σ_{n-1} is infinite
- σ_{n-1} has infinitely many infinite orbits
- $F(\sigma_{n-1})$ is infinite
- $F(\sigma_j) \supseteq^* F(\sigma_{n-1})$ for $j \in n$
- if O is an infinite orbit of $F(\sigma_{n-1})$ and $j \in n-1$ then $|O \setminus F(\sigma_j)| \leq 1$

Example

A growing family $\{\sigma_i\}_{i\in n}$ *on* $\mathbb{Z}^{n+2}$ *is provided by defining* $\sigma_i \in \mathbb{S}(\mathbb{Z}^{n+2})$ *by*

$$\sigma_i(z_0,\dots z_{n+1}) = \begin{cases} (z_0,\dots z_{n+1}) & (\exists j > i+1) z_j \neq 0 \\ (z_0,\dots z_i, z_{i+1}+1, 0\dots, 0) & \text{otherwise} \end{cases}$$

In this example $F(\sigma_j) \supseteq F(\sigma_{n-1})$ for $j \in n$ rather than just $F(\sigma_j) \supseteq^* F(\sigma_{n-1})$.

Claim

If $\{\sigma_i\}_{i\in n}$ is growing then the group generated by $\{\sigma_i\}_{i\in n}$ is solvable and, hence, amenable.

Before examining the proof of the claim, suppose that $\{A_\xi\}_{\xi\in\kappa}$ is a $\subseteq^*$-tower generating a maximal ideal on ω. Without loss of generality, it may be assumed that $A_{\xi+1}\setminus A_\xi$ is infinite and $A_{\xi+1}\supset A_\xi$ for all ξ.

Given ξ let $\{a_{n,m}\}_{n\in\omega, m\in\mathbb{Z}\setminus\{0\}}$ enumerate $A_{\xi+1}\setminus A_\xi$ and let $\{a_n\}_{n\in\omega}$ enumerate A_ξ. Define the permutation θ_ξ by

$$\theta_\xi(m) = \begin{cases} m & \text{if } m\notin A_{\xi+1} \\ a_{n,k+1} & \text{if } m = a_{n,k} \text{ and } k\neq -1 \\ a_n & \text{if } m = a_{n,-1} \\ a_{n,0} & \text{if } m = a_n \end{cases}$$

Observe that if $\xi_1 < \xi_2 < \ldots < \xi_n$ then $\{\theta_{\xi_i}\}_{i=1}^n$ is a growing family of permutations. The same argument as in Foreman's construction shows that the action of the subgroup of $\mathbb{S}(\omega)$ generated by $\{\theta_\xi\}_{\xi\in\kappa}$ has a unique invariant mean.

Moreover, if every finite subset of $\{\theta_\xi\}_{\xi\in\kappa}$ generates a solvable group then the group is locally solvable, hence locally amenable and hence amenable by the Følner equivalence.

To sketch the main idea of the proof that growing families are solvable consider the example of the $\sigma_i \in \mathbb{S}(\mathbb{Z}^{n+2})$ defined by

$$\sigma_i(z_0, \ldots z_{n+1}) = \begin{cases} (z_0, \ldots z_{n+1}) & (\exists j > i+1) z_j \neq 0 \\ (z_0, \ldots z_i, z_{i+1} + 1, 0 \ldots, 0) & \text{otherwise} \end{cases}$$

and recall that this would be isomorphic to the general case except that $F(\sigma_j) \supseteq F(\sigma_{n-1})$ for $j \in n$ rather than $F(\sigma_j) \supseteq^* F(\sigma_{n-1})$. Let $\mathbb{G}_n$ be the subgroup of $\mathbb{S}(\mathbb{Z}^{n+2})$ generated by $\{\sigma_i\}_{i \in n}$.

Let $\mathbb{F}_n$ be the free group using the letters $\{\sigma_i, \sigma_i^{-1}\}_{i \in n}$ and for $w \in \mathbb{F}_n$ let $\sigma(w) \in \mathbb{S}(\mathbb{Z}^{n+2})$ be the corresponding permutation. Let $\mathbb{I}_n \subseteq \mathbb{F}_n$ be the subgroup of all words w such that $\sigma(w)$ is the identity. Of course, $\mathbb{G}_n$ is isomorphic to $\mathbb{F}_n / \mathbb{I}_n$.

Given a family $\{w_t \in \mathbb{F}_n \mid t : k \to 2\}$ define $[w_t]_{t \in 2^k}$ by induction on k.

Let $[w_t]_{t \in 2^1}$ denote the usual commutator

$$[w_0, w_1] = w_0 w_1 w_1^{-1} w_1^{-1}$$

and for $k > 1$ let $[w_t]_{t \in 2^k}$ denote

$$\left[[w_t^0]_{t \in 2^{k-1}}, [w_t^1]_{t \in 2^{k-1}}\right]$$

where $w_t^i = w_{i \frown t}$.

It will be shown that $[w_t]_{t \in 2^n} \in \mathbb{I}_n$ for every family $\{w_t\}_{t \in 2^n} \subseteq \mathbb{F}_n$. From this it follows that the derived series of $\mathbb{G}_n$ has length $n + 1$ and, hence, $\mathbb{G}_n$ is solvable.

First some notation is needed. Given a word $w \in \mathbb{F}_n$ and $j \in n$

- Let w/j be the word obtained from w by deleting all instances of σ_j and σ_j^{-1} in w.
- Let $e_j(w)$ be the sum of the exponents of σ_j occuring in w.
- Let $\overline{w}$ be the element of $\mathbb{G}_n$ corresponding to w.

The following claim is the key.

Claim

If $w \in \mathbb{F}_n$ and $e_{n-1}(w) = 0$ then for each $j \in \mathbb{Z}$ there is $w[j] \in \mathbb{F}_{n-1}$ such that $\overline{w}(j, \vec{x}) = (j, \overline{w_j}(\vec{x}))$.

To prove this proceed by induction on the length L of w. Let

$$w = \sigma_{\ell(L)}\sigma_{\ell(L-1)}\dots\sigma_{\ell(1)}$$

and suppose first that $\ell(1) \neq n-1$. In this case, if $j \neq 0$ then

$$\overline{w}(j, \vec{x}) = \overline{\sigma_{\ell(L)}\sigma_{\ell(L-1)}\dots\sigma_{\ell(2)}}(j, \vec{x})$$

and so it is possible to set $w[j] = \sigma_{\ell(L)}\sigma_{\ell(L-1)}\dots\sigma_{\ell(2)}[j]$ by the induction hypothesis. On the other hand, if $j = 0$ then $\sigma_{\ell(1)}(0, \vec{x}) = (0, \sigma_{\ell(1)}(\vec{x}))$ and it is possible to set

$$w[j] = \left(\sigma_{\ell(L)}\sigma_{\ell(L-1)}\dots\sigma_{\ell(2)}[j]\right)\sigma_{\ell(1)}$$

In the case that $\ell(1) = n - 1$ let J be the least integer such that

$$e_{n-1}(\sigma_{\ell(J)}\sigma_{\ell(J-1)} \ldots \sigma_{\ell(1)}) = 0$$

and let $\{x_i\}_{i=1}^{t}$ be an increasing enumeration of

$$\left\{ k < J \;\middle|\; \ell(k+1) \neq n-1 \text{ and } e_{n-1}(\sigma_{\ell(k)}\sigma_{\ell(k-1)} \ldots \sigma_{\ell(1)}) = -j \right\}$$

and let $w^* = \sigma_{\ell(x_t+1)}\sigma_{\ell(x_{t-1}+1)} \ldots \sigma_{\ell(x_1+1)}$ and let

$$w[j] = \left(\sigma_{\ell(L)}\sigma_{\ell(L-1)} \ldots \sigma_{\ell(J+1)}[j]\right) w^*$$

To prove the main claim proceed by induction on n noting that $\mathbb{G}_1$ is abelian to begin. For the general case, let $\{w_t\}_{t\in 2^n} \subseteq \mathbb{F}_n$ and suppose that $n > 1$. Observe that if $t : n-1 \to 2$ then $e_j([w_{t\frown 0}, w_{t\frown 1}]) = 0$ and, in particular, $e_{n-1}([w_{t\frown 0}, w_{t\frown 1}]) = 0$. For each such t and $m \in \mathbb{Z}$ let $z_t^m = [w_{t\frown 0}, w_{t\frown 1}][m]$.

The induction hypothesis yields that $[z_t^m]_{t\in 2^{n-1}} \in \mathbb{I}_{n-1}$ for each m. It follows that $[w_t]_{t\in 2^n} \in \mathbb{I}_n$ because

$$\overline{z_t^m}(m, \vec{x}) = (m, \overline{[w_{t\frown 0}, w_{t\frown 1}]}(\vec{x}))$$

This argument has left out the possibility that $F(\sigma_{n-1}) \subseteq^* F(\sigma_j)$ rather than $F(\sigma_{n-1}) \subseteq F(\sigma_j)$ that was assumed for the preceding argument. Once this is taken into account it yields the following

Theorem (Raghavan – Steprāns)

Assuming there is an ultrafilter generated by a tower, there is a subgroup $G \subseteq \mathbb{S}(\omega)$ whose action on ω has a unique invariant mean and that has a generating set all of whose elements have infinite order. The group is a solvable extension of a locally finite group and, hence, amenable.

Question

Is there a locally solvable subgroup of $\mathbb{S}(\omega)$ whose action on ω has a unique invariant mean in the Cohen model?

Question

Is there a model where there is a locally solvable (or even locally nilpotent) subgroup $G \subseteq \mathbb{S}(\omega)$ whose action on ω has a unique invariant mean?

The following is a warm-up question to the main open question:

Question

Is there a construction, not using anything more than Choice, of a subgroup of $\mathbb{S}(\omega)$ whose action on ω has a unique invariant mean and which does not contain $\mathbb{F}_2$?