Universal functions, strong colouring and PID

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Model theoretic saturation is defined in a very general context but we will look at a very restricted class of objects.

Definition

A symmetric function $F : \omega_1^2 \to \kappa$ *is (countably) saturated if for every* $X \in [\omega_1]^{\aleph_0}$ *and* $f : X \to \kappa$ *there is* $\xi \in \omega_1$ *such that* $F(\eta, \xi) = f(\eta)$ *for all* $\eta \in X$.

Definition

A symmetric function $U : \omega_1^2 \to \kappa$ *is universal if for every* $F : \omega_1^2 \to \kappa$ *there is a one-to-one function* $e : \omega_1 \to \omega_1$ *such that* $F(\xi, \eta) = U(e(\xi), e(\eta))$ *for all* $(\xi, \eta) \in \omega_1^2$.

Theorem

If $F : \omega_1^2 \to \kappa$ is saturated then it is universal.

However, there is a saturated $F : \omega_1^2 \to \kappa$ if and only if $2^{\aleph_0} = \aleph_1$. But universality seems weaker and the following theorem of Shelah shows that this is so.

Theorem (Shelah)

It is consistent with set theory that $2^{\aleph_0} = \aleph_2$ and there is a universal $U : \omega_1^2 \to 2$.

The range of U in this theorem can be replaced by ω. The original methods establish this, but it also follows from more general results of Mekler.

Question

Does the existence of a universal $U : \omega_1^2 \to 2$ *imply the existence a universal* $U : \omega_1^2 \to \omega$*?*

Since Shelah's original model for the consistency of $2^{\aleph_0} = \aleph_2$ and a universal $U : \omega_1^2 \to 2$ is a finite support iteration of ccc partial orders, the following question arises:

Question

Do cardinal invariants imply the existence or non-existence of a universal $U : \omega_1^2 \to 2$*?*

Question

Are there interesting weaker versions of a universality?

- By a tree T will be meant a subset $T \subseteq {}^{\smile\omega}\omega = \bigcup_{n\in\omega}\omega^n$ that is closed under initial segments.
- If T is a tree and $t \in T$ then $T[t]$ will denote the tree defined by

$$T[t] = \{s \in T \mid s \subseteq t \text{ or } t \subseteq s\}$$

  and $\mathbf{succ}_T(t)$ will denote the set $\{s \in T \mid s \supseteq t \text{ and } |s| = |t| + 1\}$.
- A tree T will be called *infinite splitting* if $|\mathbf{succ}_T(t)| \in \{1, \aleph_0\}$ for each $t \in T$.
- Define $\mathbf{split}(T) = \{t \in T \mid |\mathbf{succ}_T(t)| = \aleph_0\}$ and define

$$\mathbf{split}_n(T) = \{t \in \mathbf{split}(T) \mid |\{k \in |t| \mid t \restriction k \in \mathbf{split}(T)\}| = n\}.$$

The partial orders

Definition

Miller forcing, is denoted by **PT** *and consists of all infinite splitting trees ordered by inclusion.*

Definition

Laver forcing, on the other hand, is denoted by **LT** *consists of all infinite splitting trees such that* $T \setminus \mathbf{split}(T)$ *is finite, also ordered by inclusion.*

The ideal $\mathcal{S}(G)$

Definition

A function $\psi : {}^{\underset{\smile}{\omega}}\omega \to [\omega_1]^{<\aleph_0}$ *satisfying that* $\psi(s) \cap \psi(t) = \varnothing$ *unless* $s = t$ *will be said to have* disjoint range. *If* G *is a filter of subtrees of* ${}^{\underset{\smile}{\omega}}\omega$ *and* ψ *has disjoint range define*

$$S(G, \psi) = \bigcup_{t \in \bigcap G} \psi(t).$$

If G *is a generic filter of trees over a model* V *define*

$$\mathcal{S}(G) = \{S(G, \psi) \mid \psi \in V \text{ and } \psi \text{ has disjoint range}\}.$$

It will be shown that in various generic extensions $\mathcal{S}(G)$ is a P-ideal.

Annoying notation

- If T is infinite splitting then let $\Psi_T : {}^{\underset{\smile}{\omega}}\omega \to \mathbf{split}(T)$ be the unique bijection from ${}^{\underset{\smile}{\omega}}\omega$ to $\mathbf{split}(T)$ preserving the lexicographic ordering.
- For $t \in {}^{\underset{\smile}{\omega}}\omega$ let $T\langle t\rangle = T[\Psi_T(t)]$ Hence $\mathbf{stem}(T)$ can be defined to be $\Psi_T(\varnothing)$ for infinite splitting trees T.
- Let $\{u_i\}_{i\in\omega}$ enumerate ${}^{\underset{\smile}{\omega}}\omega$ in such a way that if $k < |u_i|$ then there is $j \in i$ such that $u_i \restriction k = u_j$.
- Then for infinite splitting trees T and S the ordering $\leq_n$ is defined by $T \leq_n S$ if $T \subseteq S$ and $\Psi_S(u_j) = \Psi_T(u_j)$ for all $j \leq n$.
- Define $T_{\setminus \mathbf{stem}}$ denote $\{t \in T \mid \mathbf{stem}(T) \subsetneq t\}$.

Lemma

If $T \Vdash_{\mathbf{PT}}$ "$\dot{S} \in \mathcal{S}(\dot{G})$ *and* $\dot{f} : \dot{S} \to 2$" *then there is* $T^* \subseteq T$ *and* $f^* : \omega_1 \to 2$ *such that*

$$T^* \Vdash_{\mathbf{PT}} \text{"}f^* \restriction \dot{S} = \dot{f}\text{"}.$$

Given $T \in \mathbf{PT}$ find $\bar{T} \subseteq T$ and ψ with disjoint range such that $\bar{T} \Vdash_{\mathbf{PT}}$ "$\dot{S} = S(\dot{G}, \psi)$". Now construct T_n and f_n^* such that:

1. $T_0 = \bar{T}$
2. $T_{n+1} \leq_n T_n$
3. the domain of f_n^* is $\bigcup_{j \in n} \bigcup_{k \leq |\Psi_{T_n}(u_j)|} \psi(\Psi_{T_n}(u_j) \restriction k)$
4. if $j \in n$ then $T_n\langle u_j \rangle \Vdash_{\mathbf{PT}}$ "$f_n^* \restriction \bigcup_{k \leq |\Psi_{T_n}(u_j)|} \psi(\Psi_{T_n}(u_j) \restriction k) \subseteq \dot{f}$".

Then if $T^* = \bigcap_{n\in\omega} T_n$ it is clear that if $f^* \supseteq \bigcup_{n\in\omega} f_n^*$ then ψ and f^* witness that T^* satisfies the lemma.

To complete the induction it suffices to note that

$$T_n\langle u_n\rangle \Vdash_{\mathbf{P}} \text{“}\psi(\Psi_{T_n}(u_n)) \subseteq \dot{S}\text{”}$$

and hence there are $T^* \subseteq T_n\langle u_n\rangle$ and f^* such that $T^* \Vdash_{\mathbf{P}}$ “$\dot{f} \restriction \psi(\Psi_{T_n}(u_n)) = f^*$”. Let $f_{n+1}^* = f_n^* \cup f^*$ and note that letting

$$T_{n+1} = (T_n \setminus (T_n\langle u_n\rangle)_{\setminus \mathbf{stem}}) \cup T^*$$

satisfies (2).

Corollary (1)

$\mathbb{T}_\omega \Vdash_{\mathbf{PT}}$ *"$\mathcal{S}(\dot{G})$ is closed under subsets".*

Lemma

$\mathbb{T}_\omega \Vdash_{\mathbf{PT}}$ *"$\mathcal{S}(\dot{G})$ is closed under finite unions."*

Given Corollary 1 it suffices to show that if

$$T \Vdash_{\mathbf{PT}} \text{“}\{\dot{X}, \dot{Y}\} \subseteq \mathcal{S}(\dot{G}) \text{ and } \dot{X} \cap \dot{Y} = \varnothing\text{”} \tag{1}$$

then there is ψ^* with disjoint range and $T^* \subseteq T$ such that $T^* \Vdash_{\mathbf{PT}}$ "$\dot{X} \cup \dot{Y} = S(\dot{G}, \psi^*)$".

Begin by finding $\tilde{T} \subseteq T$ and ψ_X and ψ_Y such that $\tilde{T} \Vdash_{\mathbf{PT}}$ "$\dot{X} = S(\dot{G}, \psi_X)$ and $\dot{Y} = S(\dot{G}, \psi_Y)$".

Now let $\psi(t) = \psi_X(t) \cup \psi_Y(s)$ and construct $\{T_n\}_{n\in\omega}$ such that

1. $T_0 = \tilde{T}$
2. $T_{n+1} \leq_n T_n$
3. $\psi(s) \cap \psi(t) = \varnothing$ if $t \in T_n$ and $s \subseteq \Psi_{T_n}(u_j)$ for some $j \leq n$ and $t \neq s$.

If this can be done then let $T^* = \bigcap_n T_n$ and $\psi^* = \psi \restriction T^*$ and observe that $T^* \Vdash_{\mathbf{PT}}$ "$\dot{X} \cup \dot{Y} = S(\dot{G}, \psi^*)$".

To complete the induction note that Hypothesis 1 implies that if $t \subsetneq s \in \tilde{T}$ then $\psi_X(t) \cap \psi_Y(s) = \varnothing$ and hence (3) holds for $n = 0$. Given T_n let

$$B = \bigcup_{j \leq n} \bigcup_{s \subseteq \Psi_{T_n}(u_j)} \psi(s)$$

and keep in mind that $B^* = \{t \in T_n \mid \psi(t) \cap B \neq \varnothing\}$ is finite and $B^* \subseteq \bigcup_{j \leq n} T_n \langle u_j \rangle_{\setminus \mathbf{stem}}$.

It is therefore possible to find $T_{n+1} \leq_n T_n$ such that $B^* \cap T_{n+1} = \varnothing$ as required.

Corollary

$\mathbb{T}_\omega \Vdash_{\mathbf{PT}}$ *"$\mathcal{S}(\dot{G})$ is an ideal."*

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Lemma (3)

If $\psi_i : {}^{\smile\omega}\omega \to [\omega_1]^{<\aleph_0}$ *have disjoint range and* $T_i \in \mathbf{PT}$ *for* $i \in \omega$ *then there are* $\bar{T}_i \leq_0 T_i$ *such that*

$$(\forall i < j < \omega)(\forall t \in (\bar{T}_i)_{\setminus \mathbf{stem}})(\forall s \in (\bar{T}_j)_{\setminus \mathbf{stem}})\ \psi_j(s) \cap \psi_i(t) = \varnothing \tag{2}$$

Lemma (4)

If $\dot{S}$ *is a* **PT***-name such that* $T \Vdash_{\mathbf{PT}}$ *"*$\dot{S} \in \mathcal{S}(\dot{G})$*" and* $k \in \omega$ *then there is* $\bar{T} \leq_k T$ *and* $\psi : \bar{T} \to [\omega_1]^{<\aleph_0}$ *with disjoint range such that* $\bar{T} \Vdash_{\mathbf{PT}}$ *"*$\dot{S} \equiv^* S(\dot{G}, \psi)$*".*

Lemma 3 applied to the finite family $\{T\langle u_i\rangle\}_{i\in k}$ implies that the general case will follow easily from the case $k=0$.

For each $t \in \mathbf{split}_1(T)$ find $T_t \subseteq T[t]$ and ψ_t with disjoint range such that $T_t \Vdash_{\mathbf{PT}}$ "$\dot{S} = S(\dot{G}, \psi_t)$".
Now apply Lemma 3 to the infinite family $\{\psi_t\}_{t\in\mathbf{split}_1(T)}$ to find $\tilde{T}_t \subseteq T_t$ such that ψ defined by

$$\psi = \bigcup_{t\in\mathbf{split}_1(T)} \psi_t \restriction (\tilde{T}_t)_{\setminus \mathbf{stem}}$$

has disjoint range. Let $\bar{T} = \bigcup_{t\in\mathbf{split}_1(T)} \tilde{T}_t$. It is immediate that ψ and $\bar{T}$ satisfy the lemma.

Lemma (5)

If $T \Vdash_{\mathbf{PT}}$ "$\{\dot{S}_n\}_{n\in\omega} \subseteq \mathcal{S}(\dot{G})$ *and* $(\forall n \neq m)\ \dot{S}_n \cap \dot{S}_m = \varnothing$" *then there are* ψ_n *with disjoint range and* $T^* \subseteq T$ *such that* $T^* \Vdash_{\mathbf{PT}}$ "$(\forall n)\ \dot{S}_n \equiv^* S(\dot{G}, \psi_n)$" *and* $\psi_n(t) \cap \psi_m(s) = \varnothing$ *for all* n *and* m *and* $s \neq t$.

Construct by induction T_n and ψ_n such that:

1. $T_0 \leq_0 T$
2. $T_{n+1} \leq_{n+1} T_n$
3. $T_n \Vdash_{\mathbf{PT}}$ "$\dot{S} \equiv^* S(\dot{G}, \psi_n)$"
4. if i, j, k and ℓ are no greater than n and $s \neq t$ and $\mathbf{stem}(T) \subsetneq s \subseteq \Psi_{T_n}(u_i)$ and $\mathbf{stem}(T) \subsetneq t \subseteq \Psi_{T_n}(u_j)$ then $\psi_k(t) \cap \psi_\ell(s) = \varnothing$
5. if $B_n = \bigcup_{i \leq n} \bigcup_{k \leq n} \bigcup_{s \subseteq \Psi_{T_n}(u_i)} \psi_k(t)$ and $t \in \bigcup_{i \in n} (T_n\langle u_i \rangle)_{\setminus \mathbf{stem}}$ and $k \in n$ then $\psi_k(t) \cap B_n = \varnothing$.

If this can be done then simply let $T^* = \bigcap_n T_n$.

Suppose that T_n and $\{\psi_i\}_{i\leq n}$ have been constructed. Use Lemma 4 to find $T_{n+1} \leq_{n+1} T_n$ and $\bar{\psi}_{n+1}$ with disjoint range such that $T_{n+1} \Vdash_{\mathbf{PT}}$ "$\dot{S} \equiv^* S(\dot{G}, \bar{\psi}_{n+1})$".

To get (5) to hold at $n+1$ simply define ψ_{n+1} by

$$\psi_{n+1}(t) = \begin{cases} \varnothing & \text{if there is } j \leq n+1 \text{ such that } t \subseteq \Psi_{T_{n+1}}(u_j) \\ \bar{\psi}_{n+1}(t) & \text{otherwise.} \end{cases}$$

To see that (4) holds for $n+1$ suppose that i, j, k and ℓ are no greater than $n+1$ and $s \neq t$ and $\mathbf{stem}(T) \subsetneq s \subseteq \Psi_{T_{n+1}}(u_i)$ and $\mathbf{stem}(T) \subsetneq t \subseteq \Psi_{T_{n+1}}(u_j)$. By the definition of ψ_{n+1} it may as well be assumed that k and ℓ are less than $n+1$. Since $T_{n+1} \leq_{n+1} T_n$ it may as well be assumed that $i < j = n$ and that $t \not\subseteq \Psi_{T_{n+1}}(u_m)$ for any $m \in n$. In other words,

$$t \notin \bigcup_{i \in n} (T_n\langle u_i \rangle)_{\setminus \mathbf{stem}}$$

and hence induction hypothesis (5) implies that $\psi_k(t) \cap \psi_\ell(s) = \varnothing$ as required.

Lemma

$\mathbb{T}_\omega \Vdash_{\mathbf{PT}}$ *"$\mathcal{S}(\dot{G})$ is a P-ideal".*

It suffices to show that if

$$T \Vdash_{\mathbf{PT}} \text{"}\{\dot{S}_n\}_{n\in\omega} \subseteq \mathcal{S}(\dot{G}) \text{ and } (\forall n \neq m)\ \dot{S}_n \cap \dot{S}_m = \varnothing\text{"}$$

then there is $T^* \subseteq T$ and ψ with disjoint range such that $T^* \Vdash_{\mathbf{PT}}$ "$\bigcup_{n\in\omega} \dot{S} \subseteq S(\dot{G}, \psi)$".

Using Lemma 5 there are ψ_n with disjoint range and $T^* \subseteq T$ such that $T^* \Vdash_{\mathbf{PT}}$ "$(\forall n)\ \dot{S}_n \equiv^* S(\dot{G}, \psi_n)$" and $\psi_n(t) \cap \psi_m(s) = \varnothing$ for all n and m and $s \neq t$. Then define $\psi(t) = \bigcup_{j\leq|t|} \psi_k(t)$ and note that $T^* \Vdash_{\mathbf{PT}}$ "$(\forall n)\ \dot{S}_n \equiv^* S(\dot{G}, \psi_n) \subseteq^* S(\dot{G}, \psi)$".

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Lemma

If G is **PT** *generic over V and $S \in \mathcal{S}(G)$, $S \subseteq \xi \in \omega_1$, $f : S \to 2$ are in $V[G]$ and $Z \subseteq 2^\xi$ is nowhere meagre in V, then there is $z \in Z$ such that $f \subseteq z$.*

Let $\dot{S}$ and $\dot{f}$ be **PT**-names for S and f so that $T \Vdash_{\mathbf{PT}}$ "$\dot{S} \in \mathcal{S}(\dot{G})$ and $\dot{f} : \dot{S} \to 2$". Then find $T^* \subseteq T$, ψ and $f^* : \omega_1 \to 2$ such that $T^* \Vdash_{\mathbf{PT}}$ "$f^* \restriction \dot{S} = \dot{f}$ and $\dot{S} = S(\dot{G}, \psi)$". Note that it follows that if $t \in T^*$ then $T^*[t] \Vdash_{\mathbf{PT}}$ "$\dot{f} \restriction \psi(t) = f^* \restriction \psi(t)$".

Let $\bar{f} = \bigcup_{j \leq |\mathbf{stem}(T^*)|} f^* \restriction \psi(\mathbf{stem}(T^*) \restriction j)$ and let $\mathcal{O}$ be the open set $\left\{ h \in 2^\xi \mid h \supseteq \bar{f} \right\}$. Then $Z \cap \mathcal{O}$ is not meagre in $\mathcal{O}$. For $s \in T^*$ let s^* be the least element of $\mathbf{split}(T^*)$ such that $s \subseteq s^*$ and define

$$\mathcal{O}_s = \left\{ x \in \mathcal{O} \;\middle|\; x \supseteq \bigcup_{s \subseteq u \subseteq s^*} f^* \restriction \psi(u) \right\}.$$

If $t \in \mathbf{split}(T^*)$ then define

$$\mathcal{O}_t^+ = \bigcap_{F \in [\mathbf{succ}_{T^*}(t)]^{<\aleph_0}} \bigcup_{s \in \mathbf{succ}_{T^*}(t) \setminus F} \mathcal{O}_s$$

and note that $\mathcal{O}_t^+$ is a dense G_δ in $\mathcal{O}$ for each $t \in \mathbf{split}(T^*)$.

Hence there is some $z \in Z \cap \bigcap_{t \in \mathbf{split}(T^*)} \mathcal{O}_t^+$ such that $\bar{f} \subseteq z$ and

$(\forall k \in \omega)(\forall t \in \mathbf{split}_k(T^*))(\exists^\infty s \in \mathbf{succ}_{T^*}(t)) \bigcup_{s \subseteq u \subseteq s^*} f^* \restriction \psi(u) \subseteq z$.

It follows that there is $T^{**} \subseteq T^*$ such that $T^{**} \in \mathbf{PT}$ and such that if $t \in T^{**}$ then $f^* \restriction \psi(t) \subseteq z$ and hence $T^{**} \Vdash_{\mathbf{PT}}$ "$\dot{f} \subseteq z$".

Theorem (Abraham & Todorcevic)

Let $\mathcal{I}$ be a P-ideal on ω_1 that is generated by a family of $\aleph_1$ countable sets — in particular, this will hold if $2^{\aleph_0} = \aleph_1$ and $\mathcal{I} \subseteq [\omega_1]^{\leq\aleph_0}$. Then there is a proper partial order $\mathbb{P}_{\mathcal{I}}$, that adds no reals, even when iterated with countable support, such that there is a $\mathbb{P}_{\mathcal{I}}$-name $\dot{Z}$ for a subset of ω_1 such that for any $Y \subseteq \omega_1$ which is not the union of countably many sets orthogonal to $\mathcal{I}$

$$1 \Vdash_{\mathbb{P}_{\mathcal{I}}} \text{“}\dot{Z} \cap Y \neq \varnothing\text{”} \tag{3}$$

$$1 \Vdash_{\mathbb{P}_{\mathcal{I}}} \text{“}(\forall \eta \in \omega_1)\ \dot{Z} \cap \eta \in \mathcal{I}\text{”}. \tag{4}$$

Lemma

If G is **PT** *generic over V then no uncountable subset of ω_1 in V is orthogonal to $\mathcal{S}(G)$ in $V[G]$.*

Suppose that $\dot{Z}$ is a **PT**-name such that $T \Vdash_{\mathbf{PT}}$ "$\dot{Z} \in [\omega_1]^{\aleph_1}$". It suffices to construct a sequence of conditions $T_n \in \mathbf{PT}$ and ordinals ζ_n such that:

- $T_0 = T$,
- $T_{n+1} \leq_n T_n$ for each n
- $\zeta_n < \zeta_{n+1}$
- $T_n\langle u_j \rangle \Vdash_{\mathbf{PT}}$ "$\zeta_j \in \dot{Z}$" for each $j \in n$

because then it is possible to define $T_\omega = \bigcap_{n \in \omega} T_n$ and to define $\psi : \overset{\omega}{\smile}\omega \to \omega_1$ by

$$\psi(t) = \begin{cases} \{\zeta_j\} & \text{if } t = \Psi_{T_\omega}(u_j) \\ \varnothing & \text{otherwise.} \end{cases}$$

It is then immediate that $T_\omega \Vdash_{\mathbf{PT}}$ "$|S(\dot{G}, \psi) \cap \dot{Z}| = \aleph_0$".

To carry out the construction let T_n be given and let $\eta = \max_{j\in n} \zeta_{u_j}$. Using that $T \Vdash_{\mathbf{PT}}$ "$\dot{Z} \setminus \eta \neq \varnothing$" it is possible to find $T^* \subseteq T_n\langle u_n\rangle$ and $\zeta_n > \eta$ such that $T^* \Vdash_{\mathbf{PT}}$ "$\zeta_n \in \dot{Z}$". Let $T_{n+1} = (T_n \setminus T_n\langle u_n\rangle) \cup T^*$.

Theorem

Let V be a model of the Continuum Hypothesis and suppose that $U : \omega_1^2 \to 2$ is a symmetric, category saturated function in V and that $G \subseteq \mathbf{PT}$ is generic over V. In $V[G]$ let $H \subseteq \mathbb{P}_{\mathcal{S}(\dot{G})}$ be generic over $V[G]$. Then in $V[G][H]$ the function U is universal.

In $V[G]$ there is $R \subseteq \omega_1$ such that $[R]^{\aleph_0} \subseteq \mathcal{S}(\dot{G})$ and $R \cap Y \neq \varnothing$ for each $Y \in V[G]$ that is an uncountable subset of ω_1. Given $W : \omega_1^2 \to 2$ in $V[G][H]$ that is symmetric, construct by induction one-to-one embeddings $e_\eta : \eta \to R$ of $W \restriction \eta^2$ into U such that $e_\eta \subseteq e_\zeta$ if $\eta \leq \zeta$.

- Limit stages of the induction are trivial.
- So, given e_η let $S \subseteq \xi$ be the range of e_η.
- Then $S \in [R]^{\aleph_0} \subseteq \mathcal{S}(\dot{G})$.
- Let $f : S \to 2$ be defined by $f(\sigma) = W(e_\eta^{-1}(\sigma), \eta)$ and note that $f \in V[G]$ since $V[G]$ and $V[G][H]$ have the same reals.
- **PT** preserves non-meagre sets.
- Therefore $\{\gamma \in \omega_1 \mid f \subseteq U(*, \gamma)\}$ is an uncountable set in $V[G]$.
- There is $\gamma \in R \setminus \xi$ such that $f \subseteq U(*, \gamma)$ and, hence, $W(e_\eta^{-1}(\sigma), \eta) = f(\sigma) = U(\sigma, \gamma)$ for all $\sigma \in S$.
- Let $e_{\eta+1} = e_\eta \cup \{(\eta, \gamma)\}$.

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Corollary

Given any regular $\kappa > \aleph_1$ it is consistent with set theory that

- $\mathfrak{b} = \aleph_1$ *(indeed,* $\mathfrak{a} = \aleph_1$*)*
- $\mathfrak{d} = \aleph_2$
- $2^{\aleph_1} = \kappa$
- *there is a universal graph on* ω_1.

- The required model is the one obtained by starting with a model of the Continuum Hypothesis in which $2^{\aleph_1} = \kappa$ and iterating, with countable support, ω_2 Miller reals at even coordinates and forcing with $\mathbb{P}_{\mathcal{S}(\dot{G})}$ at odd coordinates.
- Any category saturated graph in the initial model will be universal in the final extension. To see this, begin by observing that **PT** preserves $\sqsubseteq^{\textbf{Cohen}}$.
- Since $\mathbb{P}_{\mathcal{S}(\dot{G})}$ is proper and adds no new reals it is immediate that it also preserves $\sqsubseteq^{\textbf{Cohen}}$. It follows that the entire countable support iteration preserves non-meagre sets and, hence, any category saturated graph in the initial model remains category saturated.

- To see that all of these graphs are universal use the $\aleph_2$-pic to conclude that the iteration has the $\aleph_2$ chain condition and, hence, that any graph on ω_1 appears at some stage.
- Note that bookkeeping using $2^{\aleph_1} = \aleph_2$ is not needed.
- That $\mathfrak{d} = \aleph_2$ is a standard argument using that Miller forcing adds an unbounded real.
- The fact that $\mathfrak{b} = \aleph_1$ follows from the fact that $\mathfrak{b} \leq \mathbf{non}(\mathcal{M})$.
- To see that, in fact, the stronger result $\mathfrak{a} = \aleph_1$ holds, it is not possible to use the argument of Spinas or Eisworth because $\mathbb{P}_{\mathcal{S}(\dot{G})}$ is not a Souslin forcing, indeed, it does not even have cardinality $2^{\aleph_0}$. But, an earlier argument similar to the proof that the iteration of proper partial orders is proper works.

A DIGRESSION ON STRONG COLOURINGS

QUESTION

Recall Todorcevic's theorem — improving earlier work of Sierpiński's, Galvin, Shelah — that there is a colouring $c : [\omega_1]^2 \to \omega_1$ with the property that the image of c on $[A]^2$ is all of ω_1 for all uncountable $A \subseteq \omega_1$. The existence of such a colouring is denoted by $\aleph_1 \not\to [\aleph_1]^2_{\aleph_1}$. How can this be strengthened?

One answer is provided by Moore who constructed a colouring $c : [\omega_1]^2 \to \omega_1$ with the property that the image of c on $A \circledast B = \omega_1$ for all uncountable A and B where $A \circledast B$ stands for the *rectangle* $\{(\alpha, \beta) \in A \times B \mid \alpha < \beta\}$.

A related, but somewhat different question is the following:

QUESTION (ERDÖS-GALVIN-HAJNAL)

Given $G \subseteq [\omega_1]^2$ with uncountable chromatic number, is there $c : G \to \omega_1$ such that for all $w : \omega_1 \to \omega$ there is $n \in \omega$ such that the image of c on $G \cap [w^{-1}\{n\}]^2$ is all of ω_1?

Note that the partition w is of singletons. The following definition is more in keeping with the spirit of Moore's result.

DEFINITION

Let $p : [\omega_1]^2 \to \omega$. Define $\aleph_1 \not\to_p [\aleph_1]^2_\kappa$ to mean that there is some $c : [\omega_1]^2 \to \kappa$ such that for each uncountable $X \subseteq \omega_1$ there is $n \in \omega$ such that the image of c on $p^{-1}\{n\} \cap [X]^2$ is all of κ. I will focus on $\kappa = \aleph_0$.

- It is shown in **Chen, Kojman, S.** and later in **Kojman, Rinot, S.** that it is consistent with various versions of set theory that $\aleph_1 \not\rightarrow_p [\aleph_1]^2_\kappa$ holds.
- The positive relation $\aleph_1 \rightarrow_p [\aleph_1]^2_\kappa$ is also interesting and consistent.
- One might also ask for less than $\aleph_1 \not\rightarrow_p [\aleph_1]^2_{\aleph_0}$. For example only that there is some $c : [\omega_1]^2 \rightarrow \omega$ such that for each uncountable $X \subseteq \omega_1$ there is $n \in \omega$ such that the image of c on $p^{-1}\{n\} \cap [X]^2$ is infinite, rather than all of ω.
- Even this weaker version can fail.

Lemma

If $p : [\omega_1]^2 \to \omega$ is category saturated and $c : [\omega_1]^2 \to \omega$ and $G \subseteq \mathbf{PT}$ is generic over V and $M_G = \bigcap G$ and $H \subseteq \mathbb{P}_{\mathcal{S}(\dot{G})}$ is generic over $V[G]$ and there is an uncountable $X \subseteq \omega_1$ in $V[G][H]$ such that $M_G(p(\alpha,\beta)) > c(\alpha,\beta)$ for all $\{\alpha,\beta\} \in [X]^2$.

There is in $V[G][H]$ an uncountable R such that $[R]^{\aleph_0} \subseteq \mathcal{S}(G)$ and such that $R \cap Y \neq \varnothing$ for any uncountable $Y \in V[G]$.
Construct by induction distinct $\rho_\xi \in R$ such that if $\xi \in \eta$ then

$$M_G(p(\rho_\xi,\rho_\eta)) > c(\rho_\xi,\rho_\eta).$$

To carry out the induction assume that $R_\eta = \{\rho_\xi\}_{\xi\in\eta}$ have been chosen.

Since $\mathbb{P}_{\mathcal{S}(\dot{G})}$ adds not new reals it follows that $R_\eta \in V[G]$ and so there is $T \in G$ and $\psi \in V$ with disjoint range such that $T \Vdash_{\mathbf{PT}}$ "$R_\eta = S(\dot{G}, \psi)$". Let μ be so large that $T \Vdash_{\mathbf{PT}}$ "$R_\eta \subseteq \mu$".

For each $t \in \mathbf{split}(T)$ let

$$\mathcal{W}_s = \left\{ x \in \omega^\mu \;\middle|\; x \restriction \left(\bigcup_{s \subseteq u \subseteq s^*} \psi(u) \right) \text{ has constant value } |s^*| \right\}.$$

and define

$$\mathcal{W}_t^+ = \{ x \in \omega^\mu \mid (\exists^\infty s \in \mathbf{succ}_T(t))\; x \in \mathcal{W}_s \}$$

and note that $\mathcal{W}_t^+$ is a dense G_δ in ω^μ for each $t \in \mathbf{split}(T)$.

Since p is category saturated,

$$\left\{ \beta \in \omega_1 \;\middle|\; p^\beta \in \bigcap_{t \in \mathbf{split}(T)} \mathcal{W}_t^+ \right\}$$

is uncountable.

Therefore there is some $\beta \in R \setminus R_\eta$ such that for all $t \in \mathbf{split}(T)$ there are infinitely many $s \in \mathbf{succ}_T(t)$ such that $p(\alpha, \beta) = |s^*|$ for all $\alpha \in \bigcup_{s \subseteq u \subseteq s^*} \psi(u)$.

Using this it is possible to find a tree $T^* \subseteq T$ such that

$$(\forall k)\ \mathbf{split}_k(T^*) \subseteq \mathbf{split}(T) \tag{5}$$

and such that for all $t \in \mathbf{split}(T^*)$ and all $s \in \mathbf{succ}_{T^*}(t)$

$$\left(\forall \alpha \in \bigcup_{s \subseteq u \subseteq s^*} \psi(u)\right)\ p(\alpha, \beta) = |s^*|$$

noting that s^* calculated in T is the same as in T^* because of (5).

By removing only finitely many elements of $\mathbf{succ}_{T^*}(t)$ for each $t \in \mathbf{split}(T^*)$ it is possible to find $T^{**} \subseteq T^*$ such that

$$(\forall k)\ \mathbf{split}_k(T^*) \subseteq \mathbf{split}(T) \tag{6}$$

and such that

$$(\forall t \in \mathbf{split}(T^{**}))(\forall s \in \mathbf{succ}_{T^{**}}(t))(\forall \bar{s} \in \mathbf{succ}_{T^{**}}(s^*))$$
$$\left(\forall \{\alpha, \beta\} \in \left[\bigcup_{s \subseteq u \subseteq s^*} \psi(u)\right]^2\right)\ \bar{s}(|s^*|) = \bar{s}(p(\alpha, \beta)) > c(\alpha, \beta) \tag{7}$$

noting again that s^* calculated in T^* is the same as in T^{**} because of (6). This implies that

$$M_G(p(\alpha, \beta)) > c(\alpha, \beta)$$

for all $\alpha \in R_\eta$.

- The arguments using Miller reals can be applied to the case of Laver reals, but some changes are needed.
- The notion of disjoint range has to be replaced by: A function $\psi : {}^{\underset{\smile}{\omega}}\omega \to [\omega_1]^{<\aleph_0}$ will be said to have *bounded, disjoint range* provided that:
  - $\psi(s) \cap \psi(t) = \varnothing$ unless $s = t$
  - for all $t \in {}^{\underset{\smile}{\omega}}\omega$ there is B such that $|\psi(t^\frown j)| < B$ for all j

  The definitions of $S(G, \psi)$ and $\mathcal{S}(G)$ do not change.
- Instead of of starting with a category saturated graph, start with a measure saturated graph.
- Use that Laver forcing preserves outer measure.

- In the model obtained by iterating Laver and PID forcing there is a (measure) universal graph of cardinality $\aleph_1$.
- Unlike Shelah's original model for a universal graph of cardinality $\aleph_1$, the value of $\mathfrak{b}$ is $\aleph_1$.
- One also has $\mathfrak{h} = \mathfrak{t} = \mathfrak{s} = \aleph_1$
- Of course $\mathbf{non}(\mathcal{L}) = \aleph_1 \ldots$
- ...and the Borel Conjecture holds.

Two questions have not been answered so far:

- Does the existence of a universal graph of cardinality $\aleph_1$ imply that $\mathfrak{d} > \aleph_1$?
- What about a universal function from $[\omega_1]^2 \to \omega$?

To show that the existence of a universal graph does not imply the existence of a universal function with range ω the following lemma is key.

Lemma (Shelah)

If $\mathfrak{b} = \aleph_1$ and there is a sequence of pairs of natural numbers $\{(m_i, n_i)\}_{i\in\omega}$ such that $m_i < n_i < m_{i+1}$ for each $i \in \omega$ and

$$\left(\forall \mathcal{F} \subseteq \left[\prod_{i\in\omega} [n_i]^{m_i}\right]^{\aleph_1}\right)\left(\exists g \in \prod_{i\in\omega} n_i\right)(\forall f \in \mathcal{F})\,(\forall^\infty k)\; g(k) \notin f(k) \tag{8}$$

then there is no universal function $c : [\omega_1]^2 \to \omega$.

To prove this, let $B_\eta : \eta \to \omega$ be a bijection for each $\eta \in \omega_1$. Suppose that $c : \omega_1^2 \to \omega$ is a universal universal function. If $\eta \in \xi \in \omega_1$ and $j \in \omega$ let

$$f_{\eta,\xi}(j) = \left\{ c(B_\eta^{-1}(k), \xi) \in n_j \mid k \in m_j \right\}$$

and use the hypothesis of the lemma to find a function $g_\eta \in \prod_{i\in\omega} n_i$ such that $g_\eta(j) \notin f_{\eta,\xi}(j)$ for every $\xi \in \omega_1$ and for all but finitely many $j \in \omega$.

Let $\mathcal{U}$ be a family of increasing functions from ω to ω that is $\leq^*$ unbounded and such that $|\mathcal{U}| = \aleph_1$. Let $\psi : \mathcal{U} \times \omega_1 \to \omega_1$ be a bijection and define

$$b : \omega \times \omega_1 \to \omega$$

by $b(j, \psi(u, \eta)) = g_\eta(u(j))$.

Now suppose that $e : \omega_1 \to \omega_1$ is an embedding of the partial function b into c. Let η be such that $e(j) \in \eta$ for all $j \in \omega$ and let $u \in \mathcal{U}$ be such that there are infinitely many k such that $B_\eta(e(k)) \in m_{u(k)}$.

Then choose j so large that $g_\eta(u(j)) \notin f_{\eta, e(\psi(u,\eta))}(u(j))$ and such that $B_\eta(e(j)) \in m_{u(j)}$. Then

$$b(j, \psi(u,\eta)) = g_\eta(u(j)) \neq c(B_\eta^{-1}(B_\eta(e(j))), e(\psi(u,\eta))) \\ = c(e(j), e(\psi(u,\eta))) \quad (9)$$

contradicting that e is an embedding.

To use this lemma it suffices to find a model where $\mathfrak{d} = \aleph_1$ (thus also answering the first question) and there is a sequence $\{(m_i, n_i)\}_{i\in\omega}$ such that $m_i < n_i < m_{i+1}$ and

$$\left(\forall \mathcal{F} \subseteq \left[\prod_{i\in\omega} [n_i]^{m_i}\right]^{\aleph_1}\right)\left(\exists h \in \prod_{i\in\omega} n_i\right)(\forall f \in \mathcal{F})\,(\forall^\infty k)\, h(k) \notin f(k) \tag{10}$$

Definition

Recall that $\mathbf{PT}_{f,g}$ *consists of trees* $T \subseteq \bigcup_{n\in\omega}\prod_{i\in n} f(i)$ *such that there is a function* $r:\omega\to\omega$ *satisfying that*

- $\lim_{n\to\infty} r(n) = \infty$
- $|\mathbf{succ}_T(t)| > g(|t|, r(|t|))$ *for all* $t \in T$.

For any $T \in \mathbf{PT}_{f,g}$ *fix* $r_T:\omega\to\omega$ *witnessing that* $T \in \mathbf{PT}_{f,g}$. *The ordering on* $\mathbf{PT}_{f,g}$ *is inclusion.*

Note that letting $n_i = f(i)$ and $m_i = g(i,1)$ it is clear that forcing with $\mathbf{PT}_{f,g}$ adds a function $h \in \prod_{i\in\omega} n_i$ such that for all $f \in \prod_{i\in\omega}[n_i]^{m_i}$ there is some k such that $h(j) \notin f(j)$ for all $j > k$.

The partial order $\mathbf{PT}_{f,g}$ will be used with the functions f and g defined as follows. First let $a_n > 0$ be such that $\sum_{n=0}^{\infty} a_n < 1$. Let $g(0,0) = 1$. If $g(n,n)$ has been defined let $f(n) = \max(g(n,n), 2^n)$. Then let $g(n+1,0) = 1$ and then choose $g(n+1,k+1)$ be so large that if

- $[X_{i,j}]_{i \in g(n+1,k+1), j \in n+1}$ is a matrix of independent 2-valued random variables
- the probability that $X_{i,j} = 1$ is $1/2$
- $\varphi : g(n+1,k+1) \times (n+1) \to 2$

then the probability that

$$|\{i \in g(n+1,k+1) \mid (\forall j \in n+1)\ X_{i,j} = \varphi(j)\}| \geq g(n+1,k) \tag{11}$$

is greater than $1 - a_n / \prod_{m=0}^{n} f(m)$. It will also be required that some other inequalities hold ...

Definition

Define $\psi : {}^{\omega}\!\smile\!\omega \to [\omega_1]^{<\aleph_0}$ *to be asymptotically small with disjoint range if*

1. *if* $s \neq t$ *then* $\psi(s) \cap \psi(t) = \varnothing$
2. $\lim_{t \in T} |\psi(t)|/|t| = 0$

If $G \subseteq \mathbf{PT}_{f,g}$ *is generic over* V *and* ψ *is asymptotically small with disjoint range then the definitions of* $S(G, \psi)$ *and* $\mathcal{S}(G)$ *do not need to be changed.*

- All the lemmas that held for Miller and Laver forcing now need to be reproved.
- Once this has been done, in the model obtained by iterating $\mathbf{PT}_{f,g}$ and PID forcing there is a (measure) universal graph of cardinality $\aleph_1$.
- In this model $\mathfrak{d} = \aleph_1 = \mathbf{non}(\mathcal{L})$.
- One also has the hypothesis of the key lemma. Hence there is no universal function from $[\omega_1]^2$ to ω even though there is a universal gfraph of cardinality $\aleph_1$.

Now only one question has not been answered:

Question

Is it consistent that there is a universal function from $[\omega_1]^2 \to \omega$?

To answer this the PID idea does not seem to work and we need Shelah's idea.

Definition

Suppose that $G_0 : [\omega_1]^2 \to \omega$ and $G_1 : [\omega_1]^2 \to \omega$. Define $\mathcal{E}(G_0, G_1)$ denote the set of all finite, one-to-one functions e that are isomorphisms between $G_1 \restriction \mathbf{domain}(e)^2$ and $G_0 \restriction \mathbf{range}(e)^2$; in other words, $G_1(\eta, \zeta) = G_0(e(\eta), e(\zeta))$ for all distinct η and ζ in the domain of e.

Definition

If $G_0 : [\omega_1]^2 \to \omega$ and $G_1 : [\omega_1]^2 \to \omega$ and $T \subseteq {}^{\overset{\omega}{\smile}}\omega$ is a tree then a function $E : T \to \mathcal{E}(G_0, G_1)$ will be called good *if:*

1. *if s and t belong to T and $s \subseteq t$ then $E(s) \subseteq E(t)$*
2. *if s and t belong to T then $\mathbf{range}(E(t)) \cap \mathbf{range}(E(s)) = \mathbf{range}(E(s \wedge t))$.*

Definition

Let $\mathbb{P}$ be a tree partial order. If $G_0 : [\omega_1]^2 \to \omega$ and $G_1 : [\omega_1]^2 \to \omega$ define $\mathbb{P}_{G_0,G_1}$ to consist of triples (T, E, η) such that

1. $T \in \mathbb{P}$
2. $E : T \to \mathcal{E}(G_0, G_1)$ *is good*
3. $\eta \in \omega_1$.

If $p = (T, E, \eta) \in \mathbb{P}_{G_0,G_1}$ the notation (T^p, E^p, η^p) will be used to denote (T, E, η). Define $p \leq q$ if and only if

1. $T^p \subseteq T^q$
2. $E^p(t) = E^q(t)$ *for each* $t \in T^p$ *such that* $t \subsetneq \mathbf{stem}(T^p)$
3. $E^p(t) \supseteq E^q(t)$ *for each* $t \in T^p$ *such that* $t \supseteq \mathbf{stem}(T^p)$
4. $(\mathbf{range}(E^p(t)) \setminus \mathbf{range}(E^q(t))) \cap \eta^q = \varnothing$ *for all* $t \in T^p$
5. $\eta^p \geq \eta^q$.

Definition

If $G \subseteq \mathbb{P}_{G_0,G_1}$ is generic define $E_G : \omega_1 \to \omega_1$ by
$E_G = \bigcup_{p\in G} E(\mathbf{stem}(T^p))$.

- It is immediate that E_G is a partial embedding of G_1 into G_0.
- However, some extra requirements will be needed to guarantee that E_G is a total embedding.
- Things work out nicely for $\mathbb{P}$ being Miller forcing.
- Note that we now need to deal with bookkeeping and so $2^{\aleph_1} = \aleph_2$ in the final model.
- Why can we deal with functions, rather than graphs? E_G tells us the position of an element of $\mathcal{S}(G)$.

Questions

Question

What does MA or even PFA say about the existence of a universal graph of cardinality $\aleph_1$?

Question

Does $\mathfrak{d} > \aleph_1$ and the existence of a universal graph of cardinality $\aleph_1$ imply the existence of a universal function of cardinality $\aleph_1$?

Question

Sacks forcing does not lend itself to the approach discussed here. So, if all cardinal invariants other than $\mathfrak{c}$ are $\aleph_1$ (to be precise, the values of cardinal invariants are those of the Sacks model) is there not universal graph of cardinality $\aleph_1$?