Milliken-Taylor ultrafilters without selectives

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- What is now known as *Hindman's Theorem* was proved to establish the truth of a conjecture of Graham and Rothschild.
- It says that if the positive integers are partitioned into finitely many cells, then there is an infinite set of integers all of whose non-empty finite subsets have their sum in the same cell.
- van Douwen is credited with realizing that, assuming the Continuum Hypothesis, it is possible to construct an ultrafilter $\mathcal{U}$ such that if the positive integers are partitioned into finitely many cells, then there is $X \in \mathcal{U}$ such that that all of the non-empty finite subsets of X have a sum belonging to the same cell.
- It was noticed by van Douwen that certain ultrafilters had an even stronger property, in that they had a base consisting of all of the finite sums of some set of positive integers.

- Such ultrafilters are now known as *strongly summable ultrafilters*.
- The strongly summable ultrafilters are idempotents in $(\beta\mathbb{N}, +)$ and much more.
- The question of whether the Continuum Hypothesis is needed to construct them is also attributed to van Douwen.
- By considering the places of the non-zero digits of the binary representations of the integers, one can establish a connection between some, but not all, of the theory of $(\beta\mathbb{N}, +)$ and $(\beta[\mathbb{N}]^{<\aleph_0}, \cup)$.

Notation

The non-empty finite subsets of ω will be denoted by $\mathbb{F}$. If $A \subseteq \mathbb{F}$ consists of pairwise disjoint sets then $\mathrm{FU}(A)$ will denote the set of all unions of non-empty finite subsets of A; in other words,

$$\mathrm{FU}(A) = \left\{ \bigcup a \;\middle|\; a \in [A]^{<\aleph_0} \;\;\&\;\; a \neq \varnothing \right\}.$$

Definition

An ultrafilter on $\mathbb{F}$ will be called a *union ultrafilter* if it has a base consisting of sets of the form $\mathrm{FU}(A)$.

Notation

Define a partial order $<$ on $\mathbb{F}$ by $a < b$ if $\max(a) < \min(b)$. For $A \subseteq \mathbb{F}$ and $\kappa \leq \omega$ let $[A]^{\kappa}_{<}$ denote all sets of the form $\{a_n\}_{n \in \kappa} \subseteq A$ such that $a_n < a_{n+1}$ for all $n \in \kappa$.

Definition

An ultrafilter on $\mathbb{F}$ will be called an *ordered-union ultrafilter* if it has a base consisting of sets of the form $\mathrm{FU}(A)$ where $A \in [\mathbb{F}]^{\omega}_{<}$.

Theorem (Blass and Hindman)

Assuming the Continuum Hypothesis there are union ultrafilters that are not ordered, union ultrafilters.

Definition

An ordered-union ultrafilter $\mathcal{U}$ on $\mathbb{F}$ will be called an *stable* if it satisfies the following property: Given a sequence of sets $\{A_n\}_{n\in\omega} \subseteq \mathcal{U}$ there is a sequence $\{b_n\}_{n\in\omega} \in [\mathbb{F}]^{\omega}_{<}$ such that for each k there is soime k^* such that $\mathrm{FU}(\{b_n\}_{n\geq k^*}) \subseteq A_k$ for each $k \in \omega$.

Theorem (Blass)

For an ordered-union ultrafilter $\mathcal{H}$ the following are equivalent:

1. $\mathcal{H}$ *is stable;*
2. *if* $[\mathbb{F}]^2_< = \mathcal{A}_0 \cup \mathcal{A}_1$ *then there is* $i \in 2$ *and* $H \in \mathcal{H}$ *such that* $[H]^2_< \subseteq \mathcal{A}_i$;
3. *if* $F : \mathbb{F} \to \omega$ *then there is* $H \in \mathcal{H}$ *such that one of the following holds for all* a *and* b *in* H:
   1. $F(a) = F(b)$
   2. $F(a) = F(b)$ *if and only if* $\min(a) = \min(b)$
   3. $F(a) = F(b)$ *if and only if* $\max(a) = \max(b)$
   4. $F(a) = F(b)$ *if and only if* $\min(a) = \min(b)$ *and* $\max(a) = \max(b)$
   5. $F(a) = F(b)$ *if and only if* $a = b$.

and hence, because of (2), stable ordered-union ultrafilters are sometimes known as Milliken-Taylor ultrafilters.

Notation

Given any $A \subseteq \mathbb{F}$ define $\max(A) = \{\max(a) \mid a \in A\}$ and $\min(A) = \{\min(a) \mid a \in A\}$. For any union ultrafilter $\mathcal{U}$ on $\mathbb{F}$ define $\max(\mathcal{U}) = \{\max(A) \mid A \in \mathcal{U}\}$ and $\min(\mathcal{U}) = \{\min(A) \mid A \in \mathcal{U}\}$.

Theorem (Blass and Hindman)

If $\mathcal{U}$ is a union ultrafilter on $\mathbb{F}$ then $\max(\mathcal{U})$ and $\min(\mathcal{U})$ are both P-points.

Theorem (Blass)

If $\mathcal{U}$ is an ordered-union ultrafilter on $\mathbb{F}$ then $\max(\mathcal{U})$ and $\min(\mathcal{U})$ are RK-inequivalent selective ultrafilters.

Theorem (Blass)

Assuming $2^{\aleph_0} = \aleph_1$, for any two RK-inequivalent selective ultrafilters $\mathcal{U}$ and $\mathcal{V}$ there is a stable, ordered-union ultrafilter $\mathcal{W}$ on $\mathbb{F}$ such that $\max(\mathcal{W}) = \mathcal{U}$ and $\min(\mathcal{W}) = \mathcal{V}$.

Question (Blass)

Can the existence of stable, ordered-union ultrafilters be deduced from the existence of two non-isomorphic selective ultrafilters? Blass conjectured that it cannot.

Theorem

Blass' conjecture is true.

Notation

If $a \in \mathbb{F}$ define $a^- = a \setminus \{\max a\}$. If $H \subseteq \mathbb{F}$ and $m \leq k$ define

$$H[m,k] = \left\{h^- \mid h \in H \ \& \ \max(h) = k \ \& \ \min(h) \geq m\right\}.$$

and define $H[m,\infty] = \{h \in H \mid \min(h) \geq m\}$.

Notation

Let $\mathbb{T}_n = \prod_{k \in n} 2^{\mathcal{P}(k)}$ and $\mathbb{T} = \bigcup_{n \in \omega} \mathbb{T}_n$. For $t \in T[n]$ note that

$$\mathrm{succ}_T(t) = \{f : \mathcal{P}(n) \to 2 \mid t^\frown f \in T\}.$$

Let $\mathcal{H}$ be a stable, ordered-union ultrafilter. Define the partial order $\mathbb{P}(\mathcal{H})$ to consist of trees $T \subseteq \mathbb{T}$ such that there is $H \in \mathcal{H}$ such that for each $\ell \in \omega$ and for all but finitely many $k \in \max(H)$

$$(\forall t \in T[k])(\forall g : H[\ell, k] \to 2^{\mathcal{P}(\ell)})(\exists f \in \text{succ}_T(t)) \\ (\forall x \subseteq \ell)(\forall h \in H[\ell, k])\ f(x \cup h) = g(h)(x). \quad (1)$$

Observe that if $f : \mathcal{P}(n) \to 2$ and $f * \ell : \mathcal{P}([\ell, n)) \to 2^{\mathcal{P}(\ell)}$ is defined by $f * \ell(h)(x) = f(x \cup h)$ then (1) is equivalent to

$$(\forall t \in T[k])(\forall g : H[\ell, k] \to 2^{\mathcal{P}(\ell)})(\exists f \in \text{succ}_T(t))\ g \subseteq f * \ell.$$

The ordering on $\mathbb{P}(\mathcal{H})$ is inclusion. If $G \subseteq \mathbb{P}(\mathcal{H})$ is generic then let B_G be the generic branch of T and define $\mathbb{C}_G : \mathbb{F} \to 2$ by $\mathbb{C}_G(a) = \bigcup_{t \in B_G} t(\max(a))(a^-)$.

Definition

Let $\mathcal{H}$ be a stable ordered-union ultrafilter. Define the game $\mathbb{G}(\mathcal{H})$ as follows. At stage k of the game Player 1 plays $S_k \in \mathcal{H}$. Then Player 2 plays $a_k \in S_k$. The play of the game is won by Player 2 if $A \in [\mathbb{F}]^{\omega}_{<}$ and $\mathrm{FU}(A) \in \mathcal{H}$.

It is important to note that the game $\mathbb{G}(\mathcal{H})$ is not played by having Player 1 play $\{b_k\}_{k\in\omega}$ such that $\mathrm{FU}(\{b_k\}_{k\in\omega}) \in \mathcal{H}$ and then having Player 2 play some b_k. (This is a stronger property called *sparesness*.)

Lemma

If $\mathcal{H}$ is a stable, ordered-union ultrafilter then Player 1 *has no winning strategy in the game $\mathbb{G}(\mathcal{H})$.*

Suppose that Σ is a strategy for Player 1; in other words, $\Sigma(\sigma) \in \mathcal{H}$ for each sequence $\sigma \in [\mathbb{F}]^{<\omega}_{<}$. For $k \in \omega$ let

$$\Sigma^*(k) = \bigcap \left\{ \Sigma(\sigma) \;\middle|\; \sigma \in [\mathcal{P}(k)]^{\leq k}_{<} \right\}$$

and define a partition $[\mathbb{F}]^2_< = \mathcal{A}_0 \cup \mathcal{A}_1$ by letting $\{a, b\} \in \mathcal{A}_0$ if and only if $b \in \Sigma^*(\max(a))$. Using the equivalent condition of Blass' theorem it is possible to find $H \in \mathcal{H}$ which is homogeneous for this partition and note that it can only be the case that $[H]^2_< \subseteq \mathcal{A}_0$. Let $A = \{a_i\}_{i\in\omega} \in [H]^\omega_<$ be such that $\mathrm{FU}(A) \in \mathcal{H}$. Then this is a winning play of the game.

Lemma

Let $\mathcal{H}$ be a stable, ordered-union ultrafilter and suppose that $T \in \mathbb{P}(\mathcal{H})$ and $D_n \subseteq \mathbb{P}(\mathcal{H})$ is dense for each $n \in \omega$. Then there is $T^* \in \mathbb{P}(\mathcal{H})$ such that

1. $T^* \subseteq T$
2. *there are infinitely many k such that $T^*\langle t\rangle \in D_k$ for each $t \in T[k]$.*

Corollary

If $\mathcal{H}$ is a stable, ordered-union ultrafilter then $\mathbb{P}(\mathcal{H})$ satisfies an appropriate continuous reading of names.

Corollary

If $\mathcal{H}$ is a stable, ordered-union ultrafilter then $\mathbb{P}(\mathcal{H})$ is

- *proper*
- *ω^ω-bounding.*

LEMMA

If $\mathcal{H}$ is a stable, ordered-union ultrafilter and $\mathbb{Q}$ is ω^ω-bounding and proper and $j \in 2$ and

$$T * q \Vdash_{\mathbb{P}(\mathcal{H})*\mathbb{Q}} \text{"}\dot{W} \subseteq \mathbb{C}_{\dot{G}}^{-1}\{j\} \ \&\ \dot{W} \text{ is closed under unions"}$$

then there is $T^* * q^* \leq T * q$ and $Z \in \mathcal{H}$ such that $T^* * q^* \Vdash_{\mathbb{P}(\mathcal{H})*\mathbb{Q}}$ "$Z \cap \dot{W} = \varnothing$".

- If $T * q \not\Vdash_{\mathbb{P}(\mathcal{H})*\mathbb{Q}}$ "$(\forall k \in \omega)(\exists w \in \dot{W})\ \min(w) > k$" then the result is immediate, so let $\dot{\psi}$ be such that

$$T * q \Vdash_{\mathbb{P}(\mathcal{H})*\mathbb{Q}} \text{"}(\forall k \in \omega)\ \dot{\psi}(k) \in \dot{W}\ \ \&\ \ \min(\dot{\psi}(k)) > k\text{"}.$$

- $\mathbb{P}(\mathcal{H}) * \mathbb{Q}$ is ω^ω-bounding, so it is possible to find a $\Psi : \omega \to \omega$ such that, without loss of generality,

$$T * q \Vdash_{\mathbb{P}(\mathcal{H})*\mathbb{Q}} \text{"}(\forall k \in \omega)\ \dot{\psi}(k) \subseteq [k+1, \Psi(k)]\text{"}.$$

- Let $B = \{b_i\}_{i\in\omega} \in [\mathbb{F}]^\omega_<$ be such that $H = \mathrm{FU(B)}$ witnesses that $T \in \mathbb{P}(\mathcal{H})$ and such that $\Psi(\max(b_n)) < \min(b_{n+1})$ for all n and, if $\ell^*_n = \Psi(\max(b_n))$, then for all $k \in \max(\mathrm{FU}(\{b_i\}_{i\geq n+1}))$ and $t \in T[k]$ and $g : H[\ell^*_n, k] \to 2^{\mathcal{P}(\ell^*_n)}$ there is $f \in \mathrm{succ}_T(t)$ such that $g \subseteq f * \ell^*_n$.

- For $t \in T[\max(b_{i+1})]$ let

$$\mathcal{S}(t) = \{f \in \mathrm{succ}_T(t) \mid (\forall x \subseteq \ell_i^*))(\forall h \in H[\ell_i^*, \max(b_{i+1})]) \text{ if } \max(x$$

- Letting $\ell_n = \max(b_n)$ it follows that if $t \in T[\max(b_{n+1})]$ and $g : H[\ell_n, \max(b_{n+1})] \to 2^{\mathcal{P}(\ell_n)}$ there is $f \in \mathcal{S}(t)$ such that $g \subseteq f * \ell_n$.
- Therefore if T^* is defined by

$$T^* = \bigcap_{i \in \omega} \left(\bigcup_{t \in T[\max(b_i)]} \bigcup_{f \in \mathcal{S}(t)} T\langle t^\frown f \rangle \right)$$

then $\mathrm{succ}_{T^*}(t) = \mathcal{S}(t)$ for each $i \in \omega$ and $t \in T^*[\max(b_i)]$.

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Last two of the three key lemmas

Notation

If $\mathcal{H}$ is a stable, ordered-union ultrafilter let $\mathcal{H}_{\text{min}}$ and $\mathcal{H}_{\text{min}}$ denote the image of $\mathcal{H}$ under min and max respectively.

Lemma

If $\mathcal{H}$ is a stable, ordered-union ultrafilter and $T \Vdash_{\mathbb{P}(\mathcal{H})}$ "$\dot{Z} \subseteq \omega$" then there is $T^* \subseteq T$ and $X \in \mathcal{H}_{\text{min}}$ such that either $T^* \Vdash_{\mathbb{P}(\mathcal{H})}$ "$X \subseteq \dot{Z}$" or $T^* \Vdash_{\mathbb{P}(\mathcal{H})}$ "$X \cap \dot{Z} = \varnothing$".

Lemma

If $\mathcal{H}$ is a stable, ordered-union ultrafilter and $T \Vdash_{\mathbb{P}(\mathcal{H})}$ "$\dot{Z} \subseteq \omega$" then there is $T^* \subseteq T$ and $X \in \mathcal{H}_{\text{max}}$ such that either $T^* \Vdash_{\mathbb{P}(\mathcal{H})}$ "$X \subseteq \dot{Z}$" or $T^* \Vdash_{\mathbb{P}(\mathcal{H})}$ "$X \cap \dot{Z} = \varnothing$".

Proof of lemma

Lemma

If $\mathcal{H}$ is a stable, ordered-union ultrafilter and $T \Vdash_{\mathbb{P}(\mathcal{H})}$ "$\dot{Z} \subseteq \omega$" then there is $T^* \subseteq T$ and $X \in \mathcal{H}_{\min}$ such that either $T^* \Vdash_{\mathbb{P}(\mathcal{H})}$ "$X \subseteq \dot{Z}$" or $T^* \Vdash_{\mathbb{P}(\mathcal{H})}$ "$X \cap \dot{Z} = \varnothing$".

Find $S \subseteq T$ and H witnessing that $S \in \mathbb{P}(\mathcal{H})$ such that for each $k \in \max(H)$ and $t \in S[k+1]$ there is $z_t \subseteq k+1$ such that $S\langle t\rangle \Vdash_{\mathbb{P}(\mathcal{H})}$ "$\dot{Z} \cap (k+1) = z_t$".
The first case to consider is that there is some $S^* \subseteq S$ and $B = \{b_i\}_{i\in\omega} \in [\mathbb{F}]^\omega_<$ such that $H = \mathrm{FU}(\{b_i\}_{i\in\omega})$ witnesses that $S^* \in \mathbb{P}(\mathcal{H})$ such that there is $A \subseteq \omega$ such that

- $\{\min(b_i) \mid i \in A\} \in \mathcal{H}_{\min}$
- for each $i \in A$ and $t \in S^*[\min(b_i)]$ there is $\tau(t) \supseteq t$ in $S^*[\max(b_i)+1]$ such that $t \subseteq \tau(t)$ and such that $\min(b_i) \in z_{\tau(t)}$.

Since $\mathcal{H}_{\max} \not\equiv_{\mathrm{RK}} \mathcal{H}_{\min}$ the mapping on $\{\min(b_i) \mid i \in A\}$ that sends $\min(b_i)$ to $\max(b_i)$ is not an RK equivalence and so there is $A^* \subseteq A$ such that

- $X = \{\min(b_i) \mid i \in A^*\} \in \mathcal{H}_{\min}$
- $Y = \{\max(b_i) \mid i \in A^*\} \notin \mathcal{H}_{\max}$.

Let T^* be defined by

$$T^* = \bigcap_{i \in A^*} \bigcup_{t \in S^*[\min(b_i)]} S^*\langle \tau(t) \rangle.$$

To see that $T^* \in \mathbb{P}(\mathcal{H})$ it suffices to verify that $\{h \in H \mid \max(h) \notin Y\}$ witnesses this. Note that if $i \in A^*$ then

$$\left(\bigcup_{t \in S^*[\min(b_i)]} S^*\langle \tau(t) \rangle \right) \Vdash_{\mathbb{P}(\mathcal{H})} \text{“}\min(b_i) \in \dot{Z}\text{”}$$

and so $T^* \Vdash_{\mathbb{P}(\mathcal{H})}$ “$X \subseteq \dot{Z}$”.

Hence it can be assumed that for every $S^* \subseteq S$ and $B = \{b_i\}_{i\in\omega} \in [\mathbb{F}]^{\omega}_{<}$ such that $H = \mathrm{FU}(\{b_i\}_{i\in\omega})$ witnesses that $S^* \in \mathbb{P}(\mathcal{H})$ there is $A \subseteq \omega$ such that $\{\min(b_i) \mid i \in A\} \in \mathcal{H}_{\min}$ and

$$(\forall i \in A)(\exists t \in S^*[\min(b_i)])(\forall \tau \in S^*\langle t\rangle[\max(b_i)+1])\ \min(b_i) \notin z_\tau.$$

Now devise a strategy for Player 1 in the game $\mathbb{G}(\mathcal{H})$. If Player 2 has played $\{b_i\}_{i\in N}$ at Inning N, then Player 1 first chooses $Y_t \in \mathcal{H}_{\min}$ for each $t \in S_N[\max(b_{N-1})+1]$ such that for each $y \in Y_t$ there is $\tau_{t,y} \in S_N\langle t\rangle[y]$ such that $S_N\langle \tau_{t,y}\rangle \Vdash_{\mathbb{P}(\mathcal{H})}$ "$y \notin \dot{Z}$". Let H_N witness that $S_N \in \mathbb{P}(\mathcal{H})$ and to also satisfy that if $\ell = \max(b_{N-1})$ then for all $k \in \max(H_N)$ and $t \in S_N[k]$ and $g : H_N[\ell, k] \to 2^{\mathcal{P}(\ell)}$ there is $f \in \mathrm{succ}_{S_N}(t)$ such that $g \subseteq f * \ell$. Player 1 then plays

$$\{h \in H_N \mid (\forall t \in S_N[\max(b_{N-1})+1])\, \min(h) \in Y_t\}$$

noting that this is a legal play of the game. If Player 2 plays b_N then Player 1 defines

$$S_{N+1} = \bigcup_{t\in S_N[\max(b_{N-1})+1]} S_N\langle \tau_{t,\min(b_N)}\rangle.$$