Universal functions, strong colouring and PID

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Strong colourings

Theorem (Todorcevic)

There is a colouring $c : [\omega_1]^2 \to \omega_1$ with the property that the image of c on $[A]^2$ is all of ω_1 for all uncountable $A \subseteq \omega_1$. The existence of such a colouring is denoted by $\aleph_1 \not\to [\aleph_1]^2_{\aleph_1}$.

This improved earlier results of Sierpiński, that $\aleph_1 \not\to (\aleph_1)^2_2$ and Galvin and Shelah, that $\aleph_1 \not\to [\aleph_1]^2_4$. If the range is understood, then such colourings are called *strong*.

Question

How can $\aleph_1 \not\to [\aleph_1]^2_{\aleph_1}$ be strengthened?

One answer is provided replacing the uncountable A by another structure

Theorem (Moore)

There is a colouring $c : [\omega_1]^2 \to \omega_1$ with the property that the image of c on $A \circledast B = \omega_1$ for all uncountable A and B where $A \circledast B$ stands for the rectangle $\{(\alpha, \beta) \in A \times B \mid \alpha < \beta\}$.

A related, but somewhat different question is the following:

Question (Erdös-Galvin-Hajnal)

Given $G \subseteq [\omega_1]^2$ with uncountable chromatic number, is there $c : G \to \omega_1$ such that for all $w : \omega_1 \to \omega$ there is $n \in \omega$ such that the image of c on $G \cap [w^{-1}\{n\}]^2$ is all of ω_1?

Definition

Let $p : [\omega_1]^2 \to \omega$. Define $\aleph_1 \not\to_p [\aleph_1]^2_\kappa$ to mean that there is some $c : [\omega_1]^2 \to \kappa$ such that for each uncountable $X \subseteq \omega_1$ there is $n \in \omega$ such that the image of c on $p^{-1}\{n\} \cap [X]^2$ is all of κ.

- If p is constant then Todorcevic's colouring shows that $\aleph_1 \not\to_p [\aleph_1]^2_\kappa$.
- In in **Chen, Kojman, S.** partitions with smaller range are considered, but this talk will not look at that case.

- It is shown in **Chen, Kojman, S.** and later in **Kojman, Rinot, S.** that it is consistent with various versions of set theory that $\aleph_1 \not\twoheadrightarrow_p [\aleph_1]^2_\kappa$ holds. For example, CH implies that $\aleph_1 \not\twoheadrightarrow_p [\aleph_1]^2_{\aleph_1}$ for any partition $p : [\omega_1]^2 \to \omega$.
- After adding $\aleph_2$ Cohen reals it is shown in [CKS] even stronger versions hold for partitions. For every partition $p : [\omega_1]^2 \to \omega$ there is a colouring $c : [\omega_1]^2 \to \omega_1$ such that for any infinite $A \subseteq \omega_1$ and uncountable $B \subseteq \omega_1$ there is $\alpha \in A$ and $n \in \omega$ such that for all $\gamma \in \omega_1$ there is $\beta \in B$ such that $c(\alpha, \beta) = \omega_1$ and $p(\alpha, \beta) = n$.
- The instance of this without a partition was shown by Todorcevic to be equivalent to a result of Sierpiński, who showed that, assuming CH, there are countably many functions $f_n : \omega_1 \to \omega_1$ such that for every uncountable $B \subseteq \omega_1$ there is some n such that $f_n(B) = \omega_1$.

Some statements under CH

1. There is a Luzin set.
2. There is an non-meagre set of size $\aleph_1$.
3. There is a sequence $\langle f_n \mid n < \omega \rangle$ of functions from from ω_1 to ω_1 such that, for every uncountable $I \subseteq \omega_1$, for all but finitely many $n < \omega$, $f_n[I] = \omega_1$.
4. There is a colouring $c : [\omega_1]^2 \to \omega_1$ such that, for all infinite $A \subseteq \omega_1$ and uncountable $B \subseteq \omega_1$, there exists $\alpha \in A$ such that $c[\{\alpha\} \times B] = \omega_1$;
5. There is a colouring $d : [\omega_1]^2 \to \omega_1$ such that, for all infinite pairwise disjoint family $\mathcal{A} \subseteq [\omega_1]^{<\aleph_0}$ and uncountable pairwise disjoint family $\mathcal{B} \subseteq [\omega_1]^{<\aleph_0}$, there exists $a \in \mathcal{A}$ such that, for every $\delta < \omega_1$, for some $b \in \mathcal{B}$, $d[a \times b] = \{\delta\}$.

- That CH implies (1) was shown by Mahlo and independently by Luzin.
- That CH implies (3) was shown by Sierpiński.
- That CH implies (4) was shown by Erdős, Hajnal and Milner.
- That CH implies (5) is due to Galvin.
- (1) implies (2) is clear, but the reverse is false.
- (5) implies (4) implies (3) are easy.
- Todorcevic showed that (1) implies (3) implies (4).
- Recently Miller showed that (2) implies (3).
- Even more recently, Guzman showed (3) implies (2).

To these equivalences Kojman and Rinot added some others, including the following:

- For every ℓ_∞-coherent partition $p : [\omega_1]^2 \to \omega$, there exists $d : [\omega_1]^2 \to \omega_1$ satisfying that given
  - an infinite pairwise disjoint subfamily $\mathcal{A} \subseteq [\omega_1]^k$ with $k < \omega$,
  - an uncountable subfamily $\mathcal{B} \subseteq [\omega_1]^l$ with $l < \omega$, such that

  there exists $a \in \mathcal{A}$ such that for every matrix $\langle \tau_{n,m} \mid n < k, m < l \rangle$ of functions from ω to ω_1, there exists $b \in \mathcal{B}$ such that for all $n < k$ and $m < l$

$$d(a(n), b(m)) = \tau_{n,m}(p(a(n), b(m)))$$

Definition

For a partition $p : [\omega_1]^2 \to \omega$ *:*

- p *has injective fibres if* $p(\alpha, \beta) \neq p(\alpha', \beta)$ *for all* $\alpha < \alpha' < \beta$
- p *has finite-to-one fibres if* $\{\alpha < \beta \mid p(\alpha, \beta) = \delta\}$ *is finite for all* $\beta < \kappa$ *and* $\delta < \mu$
- p *has almost-disjoint fibres if*

$$\{p(\alpha, \beta) \mid \alpha < \beta\} \cap \{p(\alpha, \beta') \mid \alpha < \beta\}$$

  is finite for all $\beta < \beta' < \kappa$
- p *has coherent fibres if* $\{\alpha < \beta \mid p(\alpha, \beta) \neq p(\alpha, \beta')\}$ *is finite for all* $\beta < \beta' < \kappa$*:*
- p *is* ℓ_∞*-coherent if for every* $(\beta, \beta') \in [\omega_1]^2$*, the set of integers* $\{p(\alpha, \beta) - p(\alpha, \beta') \mid \alpha < \beta\}$ *is finite.*

The $\rho_2 : [\omega_1]^2 \to \omega$ is an example of an ℓ_∞-coherent partition which does not have coherent fibres.

Lemma

There exists a partition $p : [\omega_1]^2 \to \omega$ with injective and almost-disjoint fibres.

Proposition

For every partition $p : [\omega_1]^2 \to \omega$ there exists a corresponding partition $\bar{p} : [\omega_1]^2 \to \omega$ with injective fibres such that, if one of the relations ... holds for $\bar{p}$, then it also holds for p.

Question

Can the hypothesis that p is an ℓ_∞-coherent partition be removed from the equivalence?

Theorem

It is consistent with the existence of a Luzin set that there is a partition $p : [\aleph_1]^2 \to \aleph_0$ *such that, for every colouring* $c : [\aleph_1]^2 \to \aleph_0$, *there is a decomposition* $\aleph_1 = \biguplus_{i<\omega} X_i$ *such that, for all* $i, j < \omega$,

$$c \restriction \{(\alpha, \beta) \in [X_i]^2 \mid p(\alpha, \beta) = j\} \text{ is constant.}$$

- This answers the question is a strong way.
- One might ask if the positive relation $\aleph_1 \to_p [\aleph_1]^2_{\aleph_0}$ can be weakened to ask for a colouring $c : [\omega_1]^2 \to \omega$ such that for each uncountable $X \subseteq \omega_1$ there is $n \in \omega$ such that the image of c on $p^{-1}\{n\} \cap [X]^2$ is infinite, rather than all of ω.
- Even this weaker version fails. The following lemma describes the p for which it fails.

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Lemma

The following are equivalent:

- $\mathfrak{d} = \aleph_1$
- *There exists a partition* $p : [\omega_1]^2 \to \omega$ *with injective and almost-disjoint fibres such that for every function* $h : \epsilon \to \omega$ *with* $\epsilon < \omega_1$, *there exists* $\gamma < \omega_1$, *such that for every* $b \in [\omega_1 \setminus \gamma]^{<\aleph_0}$, *there exists* $\Delta \in [\epsilon]^{<\aleph_0}$ *such that:*
  - *for all* $\alpha \in \epsilon \setminus \Delta$ *and* $\beta \in b$, $h(\alpha) < p(\alpha, \beta)$;
  - $p \restriction ((\epsilon \setminus \Delta) \times b)$ *is injective.*

Definition

Given a partition $p : [\omega_1]^2 \to \omega$ *a colouring* $c : [\omega_1]^2 \to \omega$ *will be called p-special if there is a partition* $W : \omega_1 \to \omega$ *and a function* $w : \omega \times \omega \to \omega$ *such that* $c(\alpha, \beta) = w(W(\alpha), p(\alpha, \beta))$ *if* $W(\alpha) = W(\beta)$.

Theorem

Assuming $MA_{\aleph_1}(K)$, there exists a partition $p : [\omega_1]^2 \to \omega$ such that all colourings $c : [\omega_1]^2 \to \omega$ are p-special.

Theorem

It is consistent that all of the following hold simultaneously:

- *There exists Luzin set;*
- *There exists a coherent Souslin tree;*
- *There exists a partition $p : [\omega_1]^2 \to \omega$ as in the previous slide such that all colourings $c : [\omega_1]^2 \to \omega$ are p-special.*

Definition

$\mathbb{Q}(p, c)$ consists of all triples $q = (a_q, f_q, w_q)$ satisfying all of the following:

1. $a_q \in [\omega_1]^{<\aleph_0}$;
2. $f_q : a_q \to \omega$ *is a function;*
3. w_q *is a function from a finite subset of* $\omega \times \omega$ *to* ω;
4. *for all* $(\alpha, \beta) \in [a_q]^2$, *if* $f_q(\alpha) = f_q(\beta)$, *then* $(f_q(\alpha), p(\alpha, \beta)) \in \mathbf{domain}(w_q)$ *and* $c(\alpha, \beta) = w_q(f_q(\alpha), p(\alpha, \beta))$.

For $G \subseteq \mathbb{Q}(p, c)$ let $X_{i,G} = \{\alpha < \omega_1 \mid \exists q \in G\,(f_q(\alpha) = i)\}$ and for all $i, j < \omega$ note that

$$\mathbb{1} \Vdash_{\mathbb{Q}(p,c)} \text{“}|\{c(\alpha, \beta) \mid (\alpha, \beta) \in [X_{i,\dot{G}}]^2 \text{ and } p(\alpha, \beta) = j\}| \leq 1\text{”}.$$

Lemma

For every partition $p : [\omega_1]^2 \to \omega$ *with injective and almost-disjoint fibers,* $\mathbb{Q}(p, c)$ *has Property K*

To prove this let $\{(a_\xi, f_\xi, w_\xi)\}_{\xi\in\omega_1}$ are given and assume that

- $w_\xi = w$ for all ξ
- $\{a_\xi\}_{\xi\in\omega_1}$ form a Δ-system (with empty root for simplicity)
- $a_\xi = \{a_\xi(j)\}_{j\in k}$ for all ξ
- there is $f : k \to \omega$ such that $f_\xi(a_\xi(j)) = f(j)$ for all ξ and j
- there is $p^* : k \times k \to \omega$ such that $p(a_\xi(j), a_\xi(i)) = p^*(j, i)$ for all ξ, i and j
- there is $w : k \times k \to \omega$ such that $w_\xi(f_\xi(a_\xi(j)), p(a_\xi(j), a_\xi(i))) = w(f(j), p^*(j, i))$ for all ξ, i and j.

Let $\mathfrak{M}_\xi$ be a continuous, increasing chain of countable elementary submodels of $H(\aleph_2, p, \in)$ and let $\tau_\xi = \omega_1 \cap \mathfrak{M}_\xi$. For each ξ find $\rho(\xi) \in \tau_\xi$ such that:

- if $i < j < k$ and $\rho(\xi) < \alpha, \beta < \tau_\xi$ then $p(a_\xi(j), \alpha) \neq p(a_\xi(i), \beta)$
- if $j < k$ and $\rho(\xi) < \alpha < \tau_\xi$ then $p(a_\xi(j), \alpha) \notin \mathbf{range}(f)$.

Let $\rho(\xi) = \rho$ for $\xi \in S$ with S stationary. It follows that $\{(a_{\tau_\xi}, f_{\tau_\xi}, w_{\tau_\xi})\}_{\xi \in S \setminus \rho}$ is linked.

Why? Because given $\rho < \xi < \eta$ in S the integers $p(a_{\tau_\xi}, a_{\tau_\eta})$ are all distinct and not in the range of f. Hence it is easy to extend f as needed.

The main goal now is to show that non-meagre sets are preserved by a finite support iteration. The following definition is needed for this.

Definition

For all $q \in \mathbb{Q}(p, c)$, $k < \omega$ and $z \in [\omega_1]^{<\aleph_0}$, define $q^\wedge(k, z)$ to be the triple (a, f, w) satisfying:

- $a := a_q \cup z$;
- $f : a \to \omega$ *is a function extending* f_q *and satisfying* $f(\alpha) = k + |z \cap \alpha|$ *for all* $\alpha \in a \setminus a_q$;
- $w_q := w$.

Note that $q^\wedge(k, z)$ may not be in $\mathbb{Q}(p, c)$, but it will be, provided that $k \supseteq \mathbf{range}(f_q)$.

Corollary

For every $\beta < \omega_1$, $D_\beta := \{q \in \mathbb{Q}(p, c) \mid \beta \in a_q\}$ *is dense, so that*

$$1 \Vdash_{\mathbb{Q}(p,c)} \text{“} \biguplus_{i<\omega} X_{i,\dot{G}} = \omega_1 \text{”}.$$

Definition

Let $p : [\omega_1]^2 \to \omega$ *be a partition. For any ordinal* η*, a finite-support iteration* $\{\mathbb{Q}_\xi\}_{\xi\in\eta}$ *will be called a p-iteration if* $\mathbb{Q}_0$ *is the trivial forcing, and, for each ordinal* ξ *with* $\xi + 1 < \eta$ *there is a* $\mathbb{Q}_\xi$*-name* $\mathring{c}_\xi$ *such that*

1. $\mathbb{1} \Vdash_{\mathbb{Q}_\xi}$ “$\mathring{c}_\xi \colon [\omega_1]^2 \to \omega$ *is a colouring*”,
2. $\mathbb{Q}_{\xi+1} = \mathbb{Q}_\xi * \mathbb{Q}(p, \mathring{c}_\xi)$.

Definition

Define $q \in \mathbb{Q}_\xi$ to be determined by recursion in the usual way so that a condition $q \in \mathbb{Q}_{\xi+1}$ is determined if $q \restriction \xi \Vdash_{\mathbb{Q}_\xi}$ "$q(\xi) = (a_{q,\xi}, f_{q,\xi}, w_{q,\xi})$" for an actual triple of finite sets.

Definition

For a determined condition q in the p-iteration, we say that k is sufficiently large for q iff $k \supseteq \mathbf{range}(f_{q,\xi})$ for all ξ in the support of q.

Definition

For a condition q in the p-iteration, $k < \omega$ and $z \in [\omega_1]^{<\aleph_0}$, define $q^\wedge(k,z)$ by letting $q^\wedge(k,z)(\xi) := q(\xi)^\wedge(k,z)$ for each ξ in the support of q.

Definition

A structure $\mathfrak{M}$ is said to be good for the p-iteration $\{\mathbb{Q}_\xi\}_{\xi\in\eta}$ if there is a large enough regular cardinal $\kappa > \eta$ such that all of the following hold:

- *$\mathfrak{M}$ is a countable elementary submodel of $(\mathcal{H}_\kappa, \in, \lhd_\kappa)$, where $\lhd_\kappa$ is a well-ordering of $\mathcal{H}_\kappa$;*
- *$p, \{\mathbb{Q}_\xi\}_{\xi\in\eta}$ and $\{\mathring{c}_\xi \mid \xi + 1 < \eta\}$ are in $\mathfrak{M}$.*

Definition

For any structure $\mathfrak{M}$ good for the p-iteration $\{\mathbb{Q}_\xi\}_{\xi\in\eta}$, for all $\xi \in \eta$ and a determined condition $q \in \mathbb{Q}_\xi$, we define $q^{\mathfrak{M}}$, as follows. The definition is by recursion on $\xi \in \eta$:

- *For $\xi = 0$ there is nothing to do.*
- *For any ξ such that $q^{\mathfrak{M}}$ has been defined for all determined q in $\mathbb{Q}_\xi$, given a determined condition $q \in \mathbb{Q}_{\xi+1}$, we consider two cases:*
  - *If $\xi \in \mathfrak{M}$, then let $q^{\mathfrak{M}} := (q \restriction \xi)^{\mathfrak{M}} * (a_{q,\xi} \cap \mathfrak{M}, f_{q,\xi} \cap \mathfrak{M}, w_{q,\xi})$*
  - *Otherwise, just let $q^{\mathfrak{M}} := (q \restriction \xi)^{\mathfrak{M}} * (\emptyset, \emptyset, \emptyset)$.*
- *For any limit $\xi \in \eta$, since this is a finite-support iteration, there is nothing new to define.*

- If q is determined, then, for every coordinate ξ in the support of q, $q^{\mathfrak{M}}(\xi)$ is a triple consisting of finite sets lying in $\mathfrak{M}$.
- It is important to note that $q^{\mathfrak{M}}$ may not, in general, be a condition because $q^{\mathfrak{M}} \restriction \eta$ may fail to force that $q^{\mathfrak{M}}(\eta) \in \mathbb{Q}(p, \dot{c}_\eta)$.
- Nevertheless, $(q^{\mathfrak{M}})^\wedge(k, z)$ is a well-defined object, since its definition does not depend on the $\dot{c}_\xi$'s.

Recall the lemma stated earlier and now required for the next technical lemma.

Lemma

If $\mathfrak{d} = \aleph_1$ then there is $p : [\omega_1]^2 \to \omega$ that is injective with almost-disjoint fibres and such that for every function $h : \epsilon \to \omega$ with $\epsilon < \omega_1$, there exists $\gamma < \omega_1$, such that for every $b \in [\omega_1 \setminus \gamma]^{<\aleph_0}$, there exists $\Delta \in [\epsilon]^{<\aleph_0}$ such that:

- *for all $\alpha \in \epsilon \setminus \Delta$ and $\beta \in b$, $h(\alpha) < p(\alpha, \beta)$;*
- *$p \restriction ((\epsilon \setminus \Delta) \times b)$ is injective.*

Lemma

Suppose that p is as in the previous slide and $\mathfrak{M}$ *is a structure good for the p-iteration* $\{\mathbb{Q}_\xi\}_{\xi\in\eta}$.
For all $\zeta \le \sup(\eta)$ *and a determined condition* $r \in \mathbb{Q}_\zeta$, *there is a finite set* $\bar{z} \subseteq \mathfrak{M} \cap \omega_1$ *such that:*

A: *For every* $z \in [\mathfrak{M} \cap \omega_1]^{<\aleph_0}$ *covering* $\bar{z}$, *and every integer k that is sufficiently large for r,* $(r^{\mathfrak{M}})^\wedge(k, z)$ *is in* $\mathfrak{M} \cap \mathbb{Q}_\zeta$ *and is determined;*

B: *For every* $z \in [\mathfrak{M} \cap \omega_1]^{<\aleph_0}$ *covering* $\bar{z}$, *and every integer k that is sufficiently large for r, for the condition* $\bar{r} := (r^{\mathfrak{M}})^\wedge(k, z)$ *and a condition* $q \in \mathfrak{M} \cap \mathbb{Q}_\zeta$, *if the following three requirements hold:*

1. $\mathfrak{M} \models q \le \bar{r}$ *and q is determined;*
2. *the mapping* $(\alpha, \beta) \mapsto p(\alpha, \beta)$ *is injective over* $(A_q \setminus A_{\bar{r}}) \times (A_r \setminus A_{\bar{r}})$;
3. $p(\alpha, \beta) > p(\alpha', \beta')$ *for all* $(\alpha, \beta) \in (A_q \setminus A_{\bar{r}}) \times (A_r \setminus A_{\bar{r}})$ *and* $(\alpha', \beta') \in [A_r]^2 \cup [A_q]^2$,

then $q \not\perp r$.

- Proceed by induction on $\zeta \leq \sup(\eta)$ proving **A** and **B** simultaneously.
- The case $\zeta = 0$ is immediate.
- The case $\zeta = 1$ is simple as well, but it may be instructive to consider it in detail since it gives some idea of the general proof.
- So c_0 is a colouring in the ground model and all conditions are determined.
- In this case if $r \in \mathbb{Q}_1$ then $r^{\mathfrak{M}}$ is a condition, as well.
- In general this is not the case and it is the reason **A** and **B** need to be carried along in the induction.
- It will be shown that $\bar{z} = \emptyset$ satisfies the conclusion.

- Let k be sufficiently large for r.
- We know that $(r^{\mathfrak{M}})^\wedge(k,z) \in \mathfrak{M} \cap \mathbb{Q}_1$ for any $z \in [\mathfrak{M} \cap \omega_1]^{<\aleph_0}$. Hence **A** is immediate.
- To see that **B** holds, suppose that we are given $z \in [\mathfrak{M} \cap \omega_1]^{<\aleph_0}$.
- Let $\bar{r} := (r^{\mathfrak{M}})^\wedge(k,z)$.
- We are also given a condition $q \in \mathfrak{M} \cap \mathbb{Q}_1$ satisfying requirements (1)–(3) above.

- To see that $q \not\leq r$, let $a := a_{q,0} \cup a_{r,0}$, $f := f_{q,0} \cup f_{r,0}$ and $w := w_{q,0} \cup w_{r,0}$.
- It is immediate to see that f and w are functions, $A_r = a_{r,0}$, $A_q = a_{q,0}$ and $A_q \cap A_r = A_{\bar{r}}$.
- We need to show that there exists a function w^* extending w for which (a, f, w^*) is a legitimate condition.
- For this, suppose that we are given $i, j < \omega$, $(\alpha, \beta), (\alpha', \beta') \in [a]^2$, with $f(\alpha) = f(\beta) = i = f(\alpha') = f(\beta')$ and $p(\alpha, \beta) = j = p(\alpha', \beta')$.
- It must be shown that $c_0(\alpha, \beta) = c_0(\alpha', \beta')$.

There are two cases to consider:

Case I If $(\alpha,\beta),(\alpha',\beta') \in [A_q]^2 \cup [A_r]^2$, then since w extends $w_{q,0}$ and $w_{r,0}$,
$c_0(\alpha,\beta) = w(i,j) = c_0(\alpha',\beta')$.

Case II If $(\alpha,\beta) \in [a]^2 \setminus ([A_q]^2 \cup [A_r]^2)$, then since $A_q \cap A_r = A_{\bar{r}}$ and $\alpha < \beta$, we infer that $(\alpha,\beta) \in (A_q \setminus A_{\bar{r}}) \times (A_r \setminus A_{\bar{r}})$. So, by Clause (3), $(\alpha',\beta') \in [a]^2 \setminus ([A_q]^2 \cup [A_r]^2)$, as well. Then, likewise $(\alpha',\beta') \in (A_q \setminus A_{\bar{r}}) \times (A_r \setminus A_{\bar{r}})$. Altogether, by Clause (2), $(\alpha,\beta) = (\alpha',\beta')$. In particular, $c_0(\alpha,\beta) = c_0(\alpha',\beta')$.

Lemma

Suppose:

- $p : [\omega_1]^2 \to \omega$ *is as in the previous lemma;*
- $L = \{l_\gamma\}_{\gamma \in \omega_1}$ *is a Luzin subset of* 2^ω*;*
- $\{\mathbb{Q}_\xi\}_{\xi \in \eta}$ *is a p-iteration with* $\eta > 0$ *a limit ordinal.*

Then $\mathbb{1} \Vdash_{\mathbb{Q}_\eta}$ *"L is Luzin".*

Suppose not. Then it can be assumed that there is a $\mathbb{Q}_\eta$-name $\mathring{T}$ such that

- $\mathbb{1} \Vdash_{\mathbb{Q}_\eta}$ " $\mathring{T} \subseteq 2^{<\omega}$ is a closed nowhere dense tree", and
- $\mathbb{1} \Vdash_{\mathbb{Q}_\eta}$ "$(\exists^{\aleph_1}\gamma)$ l_γ is a branch through $\mathring{T}$ ".

- It follows that there is an uncountable subset $\Gamma \subseteq \omega_1$ such that for each $\gamma \in \Gamma$ there is a determined condition $r_\gamma \in \mathbb{Q}_\eta$ such that $r_\gamma \Vdash_{\mathbb{Q}_\eta}$ "l_γ is a branch through $\mathring{T}$ ".
- It may assumed that there is a single $k < \omega$ which is sufficiently large for r_γ for all $\gamma \in \Gamma$.
- It may also be assumed that $\{A_{r_\gamma} \mid \gamma \in \Gamma\}$ forms a Δ-system with some root ρ.
- Let $\mathfrak{M}$ be a structure good for the p-iteration $\{\mathbb{Q}_\xi\}_{\xi \in \eta}$, with $\rho, \mathring{T}, \mathbb{Q}_\eta \in \mathfrak{M}$.

- For each $\gamma \in \Gamma$, let $\bar{z}_\gamma$ be given by the previous lemma with respect to r_γ and $\mathfrak{M}$.
- Fix an uncountable $\Gamma' \subseteq \Gamma$ and some $\bar{z} \in [\omega_1 \cap \mathfrak{M}]^{<\omega}$ such that $\bar{z}_\gamma = \bar{z}$ for all $\gamma \in \Gamma'$.
- By possibly shrinking further, we may assume the existence of q such that $(r_\gamma)^{\mathfrak{M}} = q$ for all $\gamma \in \Gamma'$.
- In particular, for every $z \in [\mathfrak{M} \cap \omega_1]^{<\aleph_0}$ covering $\bar{z}$, $q^\wedge(k, z) \in \mathfrak{M} \cap \mathbb{Q}_\eta$ is determined.

Let $\{\tau_n\}_{n\in\omega}$ enumerate $2^{<\omega}$. Recursively construct a sequence $\{(z_n, q_n, t_n)\}_{n\in\omega}$ such that:

- $z_0 = \bar{z} \cup \rho$;
- $q_n \leq q^\wedge(k, z_n)$ and q_n is a determined condition lying in $\mathfrak{M}$;
- $\tau_n \subseteq t_n \in 2^{<\omega}$ with $q_n \Vdash_{\mathbb{Q}_\eta}$ "$t_n \notin \dot{T}$";
- $z_{n+1} \supsetneq A_{q_n}$.

Let $\epsilon := \sup(\bigcup_{n\in\omega} A_{q_n}) + 1$. Define a function $h : \epsilon \to \omega$ by

$$h(\alpha) := \max\{k, p(\alpha', \beta') \mid (\alpha', \beta') \in [A_{q_{n+1}}]^2 \text{ and } \alpha \in A_{q_{n+1}} \setminus A_{q_n}\}.$$

Recalling the properties of p, fix $\gamma^* < \omega_1$ satisfying that for every $b \in [\omega_1 \setminus \gamma^*]^{<\aleph_0}$, there exists $\Delta \in [\epsilon]^{<\aleph_0}$ such that:

- $p \restriction ((\epsilon \setminus \Delta) \times b)$ is injective;
- for all $\alpha \in \epsilon \setminus \Delta$ and $\beta \in b$, $h(\alpha) < p(\alpha, \beta)$.

- $\Gamma^* := \{\gamma \in \Gamma' \mid \min(A_{r_\gamma} \setminus \rho) > \gamma^*\}$ is uncountable.
- For each $n < \omega$, consider the open set $U_n := \{I \in 2^\omega \mid t_n \subseteq I\}$.
- Set $W := \bigcap_{j=0}^{\infty} \bigcup_{j=n}^{\infty} U_{n+1}$.
- Then W is a dense G_δ set, so since $\{I_\gamma\}_{\gamma \in \Gamma^*}$ is Luzin, $I_\gamma \in W$ for all but countably many $\gamma \in \Gamma^*$.
- Set $b := A_{r_\gamma} \setminus \rho$, and then let $\Delta \in [\epsilon]^{<\aleph_0}$ be the corresponding set, as above.
- Fix a large enough $j < \omega$ such that $A_{q_{n+1}} \setminus A_{q_n}$ is disjoint from Δ for all $n \geq j$.

- As $l_\gamma \in W$, we may now fix some $n \geq j$ such that $l_\gamma \in U_{n+1}$.
- Denote $\bar{r} := (q^{\mathfrak{M}})^\wedge(k, z_{n+1})$.
- Then $(A_{q_{n+1}} \setminus A_{\bar{r}}) \subseteq (A_{q_{n+1}} \setminus A_{q_n}) \subseteq (\epsilon \setminus \Delta)$ and $(A_{r_\gamma} \setminus A_{\bar{r}}) \subseteq b$, and all of the following hold:
  1. $\mathfrak{M} \models q_{n+1} \leq \bar{r}$ and q_{n+1} is determined;
  2. the mapping $(\alpha, \beta) \mapsto p(\alpha, \beta)$ is injective over $(A_{q_{n+1}} \setminus A_{\bar{r}}) \times (A_{r_\gamma} \setminus A_{\bar{r}})$;
  3. $p(\alpha, \beta) > p(\alpha', \beta')$ for all $(\alpha, \beta) \in (A_{q_{n+1}} \setminus A_{\bar{r}}) \times (A_{r_\gamma} \setminus A_{\bar{r}})$ and $(\alpha', \beta') \in [A_{r_\gamma}]^2 \cup [A_{q_{n+1}}]^2$.
- Since $z_{n+1} \supseteq \bar{z}$ and $\bar{z}$ was given by the lemma, apply **B** and infer that $q_{n+1} \not\perp r_\gamma$.
- However, $q_{n+1} \Vdash_{\mathbb{Q}_\eta}$ "$t_{n+1} \notin \dot{T}$" and $r_\gamma \Vdash_{\mathbb{Q}_\eta}$ "l_γ is a branch through $\dot{T}$", contradicting the fact that $t_{n+1} \subseteq l_\gamma$.

Questions

Question

Let $p : [\omega_1]^2 \to \omega$ *be a partition. Does the following statement imply that every colouring is p-special? For every colouring c there is a partition* $\biguplus_{i<\omega} X_i = \omega_1$ *such that for all i and j the set*

$$\{ c(\alpha,\beta) \mid \{\alpha,\beta\} \in [X_i]^2 \cap p^{-1}\{j\} \}$$

is finite.

Question

Let $p : [\omega_1]^2 \to \omega$ *be a partition. Does the following statement imply that every colouring is p-special? For every colouring c there is an uncountable* $X \subseteq \omega_1$ *such that for all j the set*

$$|\{ c(\alpha,\beta) \mid \{\alpha,\beta\} \in [X]^2 \cap p^{-1}\{j\} \}| = 1.$$

Questions

Question

Are there classifications, under some set theoretic assumptions, of the $p : [\omega_1]^2 \to \omega$ such that every colouring is p-special? What happens under PFA?

Question

Are there classifications, under some set theoretic assumptions, of the $p : [\omega_1]^2 \to \omega$ such that $\aleph_1 \not\to_p [\aleph_1]^2_{\aleph_1}$?