P-points and related ultrafilters — part I

Juris Steprāns

York University

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Overview of lectures

1. **Lecture 1** will be an elementary introduction to P-points. Origins, characterizations, weaker and stronger properties. Their existence will also be examined.
2. **Lecture 2** will look at models of set theory without P-points. One model, which seems to destroy only P-points and nothing else, and another, that seems to be destroy many ultrafilters related to P-points, will be examined.
3. **Lecture 3** will look at destroying some P-points while preserving others. Many questions remain open in this area.

In 1954 Gilman and Henriksen published a paper [7] on $C(X, \mathbb{R})$, the ring of continuous functions from the completely regular topological space X to the real numbers, in which they define a point p in the space X to be a P-point if the only prime ideal consisting of functions that vanish at p is the ideal of all functions that vanish at p. They provide the following characterization of P-points.

Theorem

For every point $p \in X$, the following statements are equivalent:

- *p is a P-point.*
- *Every continuous function vanishing at p vanishes on a neighbourhood of p.*
- *Every countable intersection of neighbourhoods of p contains a neighbourhood of p.*

Definition

A **filter** $\mathcal{F}$ *on a set* X *is a subset of* $\mathcal{P}(X)$ *such that:*

1. *if* A *and* B *belong to* $\mathcal{F}$ *then* $A \cap B \in \mathcal{F}$
2. *if* $A \subseteq B$ *and* $A \in \mathcal{F}$ *then* $B \in \mathcal{F}$.

$\mathcal{I}$ *is an* **ideal** *on* X *if* $\{X \setminus A \mid A \in \mathcal{I}\}$ *is a filter on* X. *This filter will be denoted by* $\mathcal{I}^*$ *and, if* $\mathcal{F}$ *is a filter, then* $\mathcal{F}^*$ *will denote its corresponding ideal, known as the* **dual ideal**. *The notation* $\mathcal{F}^+$ *will be used to denote* $\mathcal{P}(X) \setminus \mathcal{F}^*$ *and,* $\mathcal{I}^+$ *will denote* $(\mathcal{I}^*)^+$. *The filter* $\mathcal{F}$ *will be said to be an* **ultrafilter** *on* X *if, in addition, for every* $A \subseteq X$ *either* $A \in \mathcal{F}$ *or* $X \setminus A \in \mathcal{F}$.

Proposition

A filter $\mathcal{F}$ *on* X *is an ultrafilter if and only if it is a maximal filter.*

Definition

$\beta\mathbb{N}$ is the (compact) topological space of all ultrafilters on $\mathbb{N}$ whose open sets are $A^* = \{\mathcal{U} \in \beta\mathbb{N} \mid A \in \mathcal{U}\}$. $\beta\mathbb{N} \setminus \mathbb{N}$ is the subset of $\beta\mathbb{N}$ consisting of ultrafilters that contain no finite set.

Note that the following are equivalent

$$\mathcal{U} \in A^* \cap B^*$$

$$A \in \mathcal{U} \ \ \& \ \ B \in \mathcal{U}$$

$$A \cap B \in \mathcal{U}$$

$$\mathcal{U} \in (A \cap B)^*$$

Somewhat later than Gilman and Henriksen, Walter Rudin [13] showed that $\beta\mathbb{N} \setminus \mathbb{N}$ is not homogeneous assuming $2^{\aleph_0} = \aleph_1$. His strategy was the following:

- Note that being a P-point in $\beta\mathbb{N} \setminus \mathbb{N}$ is a topological property, hence preserved by homeomorphisms.
- Show that $2^{\aleph_0} = \aleph_1$ implies that P-points exist in $\beta\mathbb{N} \setminus \mathbb{N}$.
- Show that not all points in $\beta\mathbb{N} \setminus \mathbb{N}$ are P-points.

The first part of his strategy was to prove the following

Proposition

The P-points in $\beta\mathbb{N} \setminus \mathbb{N}$ are those ultrafilters $\mathcal{U}$ satisfying: For every countable family $\mathcal{C} \subseteq \mathcal{U}$ there is $A \in \mathcal{U}$ such that $|A \setminus C| < \aleph_0$ for all $C \in \mathcal{C}$.

We will take this as our definition of a P-point. Why is it topological?

- The family $\{C^*\}_{C\in\mathcal{C}}$ is a countable family of neighbourhoods of the point $\mathcal{U}$.
- A^* is a neighbourhood of the point $\mathcal{U}$
- $A^* \cap \beta\mathbb{N} \setminus \mathbb{N} \subseteq C^*$ for all $C \in \mathcal{C}$ since $\mathcal{V} \in (A \setminus C)^*$ only if $\mathcal{V} \in \mathbb{N}$.

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We will show a bit later that much weaker hypotheses than the Continuum Hypothesis imply the existence of P-points, so we will not examine Rudin's proof here. The fact that non-P-points exist does not require any extra set theoretic hypotheses. One way of doing this goes back to Frolik [6].

Definition

If $\mathcal{F}$ is a filter on ω and $\mathcal{F}_i$ is a filter on X_i for each $i \in \omega$ then define $\prod_{\mathcal{F}} \mathcal{F}_i$ to be the filter on $\prod_{i\in\omega} X_i$ defined by

$$\prod_{\mathcal{F}} \mathcal{F}_i = \left\{ A \subseteq \prod_{i\in\omega} X_i \mid \{ i \in \omega \mid \{ x \in X_i \mid (i,x) \in A \} \in \mathcal{F}_i \} \in \mathcal{F} \right\}$$

In the special case that $\mathcal{F}_i = \mathcal{G}$ for all i where $\mathcal{G}$ is a filter on ω the notation $\mathcal{F} \times \mathcal{G}$ will be used for $\prod_{\mathcal{F}} \mathcal{F}_i$.

It is routine to verify the following.

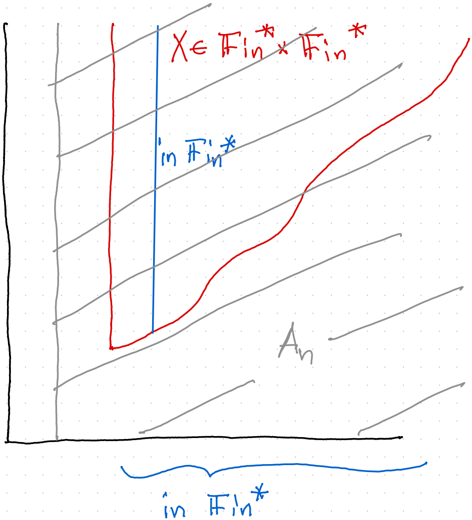
Proposition

If $\mathcal{F}$ is an ultrafilter on ω and each $\mathcal{F}_i$ is also an ultrafilter on X_i for each $i \in \omega$ then $\prod_{\mathcal{F}} \mathcal{F}_i$ is an ultrafilter on $\prod_{i\in\omega} X_i$.

The next proposition uses the product construction to establish the feature determining non-P-points, a feature that will be used without further mention throughout much of this tutorial.

Proposition

If $\mathcal{U}$ is an ultrafilter on $\omega \times \omega$ such that $\mathbb{FIN}^* \times \mathbb{FIN}^* \subseteq \mathcal{U}$ then $\mathcal{U}$ is not a P-point.


$X \in \mathbb{F}in^* \times \mathbb{F}in^*$
in $\mathbb{F}in^*$
A_n
in $\mathbb{F}in^*$

Notation

Define $A \subseteq^* B$ if $|A \setminus B| < \aleph_0$ and $A \equiv^* B$ if $A \subseteq^* B \subseteq^* A$. Let $\mathbb{FIN}$ be the ideal of finite subsets of ω.

Proof.

Let $A_n = (\omega \setminus n) \times \omega$. It follows that $A_n \in \mathbb{FIN}^* \times \mathbb{FIN}^*$ for each n and, hence $A_n \in \mathcal{U}$ for each n.

Now suppose that $A \in \mathcal{U}$ is such that $A \subseteq^* A_n$ for each n. Since $\{n\} \times \omega \cap A_{n+1} = \varnothing$ it follows that $A \cap (\{n\} \times \omega)$ is finite for each n and, hence, that $(\omega \times \omega) \setminus A \in \mathbb{FIN}^* \times \mathbb{FIN}^*$.

But this is impossible if $\mathcal{U}$ is a filter containing $\mathbb{FIN}^* \times \mathbb{FIN}^*$ and $A \in \mathcal{U}$. □

Rudin also showed that, assuming $2^{\aleph_0} = \aleph_1$, for any two P-points there is a homeomorphism of $\beta\mathbb{N} \setminus \mathbb{N}$ taking one to the other. It is important to note that here CH is essential.

Theorem (St.)

MA does not decide this.

To explain the greater generality of the result the following definition should be recalled.

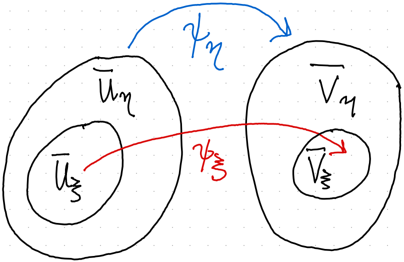
Definition

A **base** *for a filter $\mathcal{F}$ is a set $\mathcal{B} \subseteq \mathcal{F}$ such that for all $A \in \mathcal{F}$ there is $B \in \mathcal{B}$ such that $B \subseteq A$. If this holds then the filter $\mathcal{F}$ is said to be* **generated** *by $\mathcal{B}$. The character of an ultrafilter is the least cardinality of a base for it. For later use, $\mathfrak{u}$ is the least cardinality of the base of some ultrafilter.*

Proposition

If $\mathcal{U}$ and $\mathcal{V}$ are P-points and both have character $\aleph_1$ then there is a homeomorphism of Φ of $\beta\mathbb{N} \setminus \mathbb{N}$ such that $\Phi(\mathcal{U}) = \mathcal{V}$.

$\xi < \eta \in \omega_1$



$\psi_\xi \subseteq^* \psi_\eta$

Proof.

Noting that $\beta\mathbb{N} \setminus \mathbb{N}$ is the Stone space of $\mathcal{P}(\mathbb{N})/\mathbb{FIN}$ and hence, it suffices to find a Boolean algebraic isomorphism Ψ of $\mathcal{P}(\mathbb{N})/\mathbb{FIN}$ to itself such that $\Psi([A]) \subseteq \mathcal{V}$ if and only if $A \in \mathcal{U}$ where $[X]$ denotes the $\equiv^*$-equivalence class of X.

Using that both $\mathcal{U}$ and $\mathcal{V}$ are P-points of character $\aleph_1$ it is possible to find a base $\{U_\xi\}_{\xi\in\omega_1}$ for $\mathcal{U}$ and a base $\{V_\xi\}_{\xi\in\omega_1}$ for $\mathcal{V}$ such that $U_\xi \subseteq^* U_\eta$ and $V_\xi \subseteq^* V_\eta$ if $\xi \geq \eta$. Let $\bar{U}_\xi = \omega \setminus U_\xi$ and $\bar{V}_\xi = \omega \setminus V_\xi$

Next, construct by induction on ξ bijections $\psi_\xi : \bar{U}_\xi \to \bar{V}_\xi$ such that $\psi_\xi \subseteq^* \psi_\eta$ if $\xi \in \eta$. Then define Ψ as follows:

$$\Psi([A]) = \begin{cases} [\psi_\xi(A)] & \text{if } \omega \setminus A \supseteq^* U_\xi \\ [\omega \setminus \psi_\xi(\omega \setminus A)] & \text{if } A \supseteq^* U_\xi. \end{cases}$$

□

The following can be found in Booth's thesis [4].

Theorem

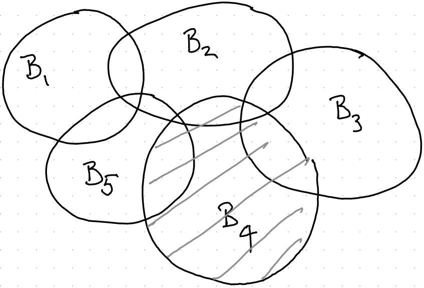
For any ultrafilter $\mathcal{U}$ the following are equivalent:

1. $\mathcal{U}$ *is a P-point.*
2. *For any compact metric space X and a sequence $\{x_n\}_{n\in\omega} \subseteq X$ there is $U \in \mathcal{U}$ such that $\{x_n\}_{n\in U}$ is a convergent sequence.*
3. *For any sequence $\{x_n\}_{n\in\omega} \subseteq 2^\omega$ there is $U \in \mathcal{U}$ such that $\{x_n\}_{n\in U}$ is a convergent sequence in 2^ω with the product topology.*

PROOF.

To see that Condition 1 implies Condition 2 let $\{x_n\}_{n\in\omega}$ be a sequence in the compact metric space X. For each n let $\mathcal{B}_n$ be finite cover of X consisting of balls of diameter less than $1/n$. It follows that there is $B_n \in \mathcal{B}_n$ such that $A_n = \{k \in \omega \mid x_k \in B_n\} \in \mathcal{U}$. Let $A \in \mathcal{U}$ be such that $A \subseteq^* A_n$ for all n. Then $\{x_n\}_{n\in A}$ is a Cauchy sequence and hence, since X is compact, this sequence converges. □

$\mathcal{B}_n$



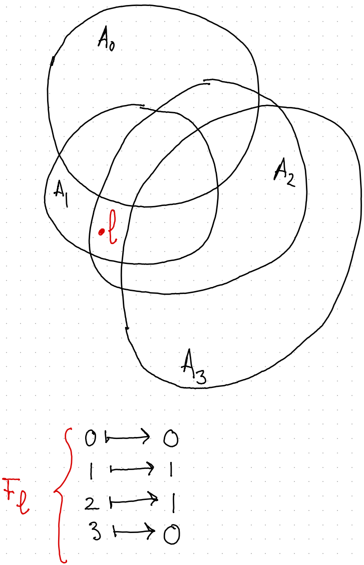
$B_n = B_4$

$\{k \mid x_k \in B_n = B_4\} \in \mathcal{U}$

Proof.

That Condition 2 implies Condition 3 is immediate.

To see that Condition 3 implies Condition 1 let $A_n \in \mathcal{U}$ for $n \in \omega$. Define $F_n : \omega \to 2$ by $F_n(m) = 1$ if and only if $n \in A_m$ and let $A \in \mathcal{U}$ be such that $\lim_{n\in A} F_n = F$ for some $F \in 2^\omega$. It remains to show that $A \subseteq^* A_m$ for each m. If this fails for some m then there are infinitely many $n \in A$ such that $F_n(m) = 0$. However, $A \cap A_m \in \mathcal{U}$ and so there are infinitley many n such that $F_n(m) = 1$. This contradicts that $\lim_{n\in A} F_n = F$. □


A0
A1
A2
A3
ℓ
Fℓ
0 ⟼ 0
1 ⟼ 1
2 ⟼ 1
3 ⟼ 0

A strikingly different characterization of P-points using an infinite game is attributed to Galvin and McKenzie by Shelah in Chapter VI of [14]. This characterization is especially useful in forcing arguments because the game can be used to simulate a generic set.

Definition

For any ultrafilter $\mathcal{U}$ *define the game* $\eth_p(\mathcal{U})$ *as follows.* Player 1 *and* Player 2 *take turns making moves for* ω *innings. At Inning* K Player 1 *plays* $A_K \in \mathcal{U}$ *and* Player 2 *plays* $a_K \in [A_K]^{<\aleph_0}$. *At the end of* ω *moves* Player 2 *is declared the winner if* $\bigcup_{k\in\omega} a_k \in \mathcal{U}$.

Lemma

If $\mathcal{U}$ *is an ultrafilter then the following are equivalent:*

1. $\mathcal{U}$ *is a P-point*
2. Player 1 *does not have a winning strategy in the game* $\eth_p(\mathcal{U})$.

To avoid technical discussions of what a strategy is, it will be convenient to rephrase this lemma using the following definition.

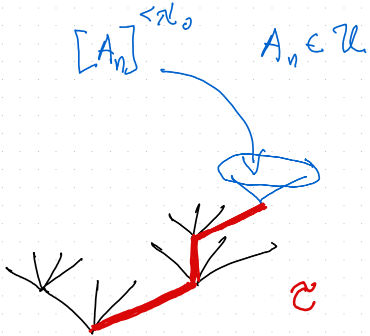
Definition

By a tree will be meant a set of finite sequences closed under initial segments; in other words, if T is a tree and $\tau \in T$ and $k \leq |\tau|$ then $\tau \restriction k \in T$. Given a tree T and $\tau \in T$ define $\mathbf{succ}_T(\tau) = \{z \mid \tau^\frown z \in T\}$. A branch of T is a function B with domain ω such that $B \restriction k \in T$ for all k.

Definition

Given an ultrafilter $\mathcal{U}$ on ω say that T is a $\mathcal{U}$-P-tree if for each $\tau \in T$ there is $A \in \mathcal{U}$ such that

- $\mathbf{succ}_T(\tau) = [A]^{<\aleph_0}$
- $\min(A) > \tau(\ell)$ *for all ℓ in the domain of τ.*


[A_n]^{<ℵ_0}
A_n ∈ 𝒰

Proposition

If $\mathcal{U}$ is an ultrafilter then $\mathcal{U}$ is a P-point if and only if every $\mathcal{U}$-P-tree has a branch B such that $\bigcup_{k\in\omega} B(k) \in \mathcal{U}$.

Proof.

Suppose first that every $\mathcal{U}$-P-tree has a branch B such that $\bigcup_{k\in\omega} B(k) \in \mathcal{U}$. Then, given $A_n \in \mathcal{U}$ it is always possible to assume that $A_{n+1} \subseteq A_n$. Then let T be the tree defined by $\mathbf{succ}_T(\tau) = [A_{|\tau|}]^{<\aleph_0}$.

For the other direction, let $\mathcal{U}$ be a P-point and T a $\mathcal{U}$-P-tree. For $\tau \in T$ let $A_\tau \in \mathcal{U}$ be such that $\mathbf{succ}_T(\tau) = [A_\tau]^{<\aleph_0}$. For $k \in \omega$ let $t(k)$ be the **finite** set of all sequences $\tau : \ell \to \mathcal{P}(k)$ for $\ell < k$ and let

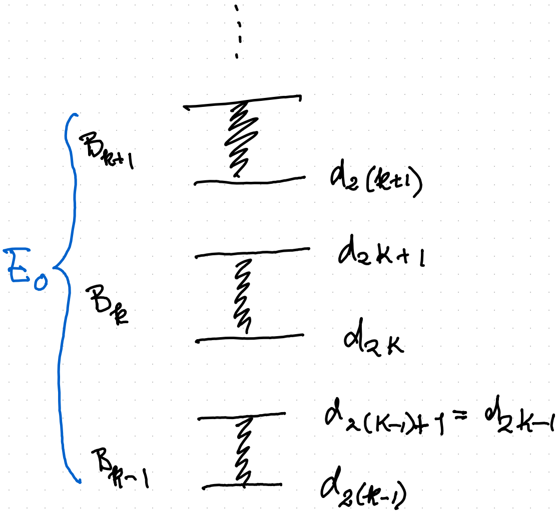
$$D_n = \bigcap_{\tau\in t(n)} A_\tau .$$

Continuation of proof.

Using that $\mathcal{U}$ is a P-point find $D \in \mathcal{U}$ such that $D \subseteq^* D_n$ for all n. Then construct a sequence of increasing integers d_n such that $d_0 = 0$ and d_{n+1} is so large that $D \setminus d_{n+1} \subseteq D_{d_n}$.

Let $E_i = \bigcup_{k\in\omega} D \cap [d_{2k+i}, d_{2k+i+1})$ for $i \in 2$. Since $D = E_0 \cup E_1$ let $j \in 2$ be such that $E_j \in \mathcal{U}$. There is no harm in letting $j = 0$. Let $B(k) = D \cap [d_{2k}, d_{2k+1})$.

□


E0
Bk+1
Bk
Bk-1
d2(k+1)
d2k+1
d2k
d2(k-1)+1 = d2k-1
d2(k-1)

Continuation of proof.

Then note that

$$B(k) \subseteq D \setminus d_{2k} \subseteq D_{d_{2k-1}} = D_{d_{2(k-1)+1}} \subseteq A_{B\restriction k}$$

To see that $D_{d_{2(k-1)+1}} \subseteq A_{B\restriction k}$ observe that if $\ell = k < 2(k-1)+1$ and $\tau(i) = B(i)$ for $i \in \ell$ then $B_i \subseteq d_{2i+1} \leq d_{2(k-1)+1}$ and so $D_{2(k-1)+1} \subseteq A_\tau = A_{B\restriction k}$.

Therefore B is a branch of T and $\bigcup_k B(k) = E_0 \in \mathcal{U}$. □

The use of games is not essential for the following, but it provides an opportunity to illustrate the concept that will be used in more subtle ways in the third lecture.

Proposition

If $\mathcal{U}$ is an ultrafilter then the following are equivalent:

1. *$\mathcal{U}$ is a P-point*
2. *for every function $F : \omega \to \omega$ there is $A \in \mathcal{U}$ such that one of the following options holds:*
    - *F is constant on A*
    - *F is finite-to-one on A.*

1 implies 2.

Suppose that F is not constant on any set on $\mathcal{U}$. Let T be the tree defined by $\varnothing \in T$ and if $\tau \in T$ then

$$\mathbf{succ}_T(\tau) = [\{n \in \omega \mid (\forall \ell \in \mathrm{domain}(\tau))\ F(n) \notin F[\tau(\ell)]\}]^{<\aleph_0}.$$

Note T is a $\mathcal{U}$-tree. Letting B be a branch of T such that $\bigcup_n B(n) \in \mathcal{U}$ yields a set in $\mathcal{U}$ on which F is finite-to-one.

□

2 implies 1.

Suppose that $\{A_n\}_{n\in\omega} \subseteq \mathcal{U}$. Define $F : \omega \to \omega + 1$ by letting $F(k)$ be the least m such that $k \notin A_m$ if there is such an m and letting $F(k) = \omega$ otherwise. If $A \in \mathcal{U}$ is such that F is constant on A then this constant value must be ω and then $A \subseteq A_m$ or all m. On the other hand, if F is finite-to-one on A, then, if $k \in A$ and $F(k) \geq m$ it follows that $k \in A_m$ and hence, $A \subseteq^* A_m$ for all m. □

Note that a consequence of the characterization of P-points in terms of convergent sequences is the following, that can be interpreted as saying that P-points are in some sense small:

Proposition

If $\mathcal{U}$ is a P-point and $F : \omega \to [0,1]$ then there is a discrete set $A \subseteq [0,1]$ such that $F^{-1}(A) \in \mathcal{U}$.

This motivates the following definition due to Baumgartner [1] but implicit in work of van Douwen.

Definition

If $\mathcal{I}$ is a family of subsets of the set X then an ultrafilter $\mathcal{U}$ is known as an $\mathcal{I}$-ultrafilter if for every function $F : \omega \to X$ there is some $A \in \mathcal{I}$ such that $F^{-1}(A) \in \mathcal{U}$.

Theorem (Baumgartner)

If $\mathcal{U}$ is an ultrafilter then in the following list of conditions on $\mathcal{U}$, each implies the next:

1. *$\mathcal{U}$ is a P-point.*
2. *$\mathcal{U}$ is a discrete-ultrafilter.*
3. *$\mathcal{U}$ is a scattered-ultrafilter.*
4. *$\mathcal{U}$ is a measure zero-ultrafilter.*
5. *$\mathcal{U}$ is a nowhere dense-ultrafilter.*

and, assuming $\mathfrak{p} = \mathfrak{c}$, none of the implications reverses.

The proof that (5) does not imply (4) is quite delicate, but even the proof that (2) does not imply (1) is illuminating.

Proof of 2 does not imply 1.

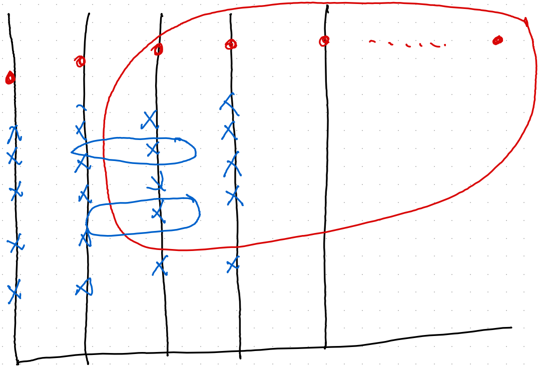
Using $\mathfrak{p} = \mathfrak{c}$ there is a P-point $\mathcal{U}$ (we will prove this soon). The non-implication will follow from showing that $\mathcal{U} \times \mathcal{U}$ is discrete. So let $F : \omega \times \omega \to [0, 1]$

For $k \in \omega$ let $F_k : \omega \to [0, 1]$ be defined by $F_k(n) = F(k, n)$. From preceding proposition get that there is are $A_k \in \mathcal{U}$ and $x_k \in [0, 1]$ such that $\lim_{n \in A_k} F_k(n) = x_k$. Let $G(k) = x_k$ and again get $A \in \mathcal{U}$ and $x \in [0, 1]$ such that $\lim_{n \in A} G(n) = x$.

Now choose disjoint neighbourhoods V_k of x_k for $k \in A$ and note that

$$D = \bigcup_{n \in A} \{F_n(m) \in V_n \mid m \in A_n\}$$

is discrete.



Proof continued.

Then $A_k \subseteq^* F_k^{-1}(V_k)$ for $k \in A$ and hence $F_k^{-1}(V_k) \in \mathcal{U}$. Therefore

$$F^{-1}(D) = \prod_{k \in A} F_k^{-1}(V_k) \in \mathcal{U} \times \mathcal{U}.$$

Since D is discrete this shows that $\mathcal{U} \times \mathcal{U}$ is a discrete-ultrafilter. Recall that $\mathcal{U} \times \mathcal{U}$ is never a P-point. □

So a nowhere dense-ultrafilters can be considered to be much weaker than P-points. Nevertheless , it will be shown in the next lecture that these may still not exist. Also studied are the ordinal ultrafilters defined for indecomposable ordinals and their associated ideals [5, 9]. All of these are notions weaker than P-point, but an important class of modifications yields stronger properties.

The following question of Baumgartner from [1] seems to be open.

Question

Does the existence of a scattered ultrafilter imply the existence of a P-point?

Definition

For any ultrafilter $\mathcal{U}$ define the game $\partial_s(\mathcal{U})$ as follows. Player 1 *and* Player 2 *take turns making moves for ω innings. At Inning K* Player 1 *plays $A_K \in \mathcal{U}$ and* Player 2 *plays $a_K \in A_K$. At the end of ω moves* Player 2 *is declared the winner if $\{a_k\}_{k\in\omega} \in \mathcal{U}$.*

The associated trees are very easy to define.

Definition

For any ultrafilter $\mathcal{U}$ define T to be a $\mathcal{U}$-S-tree if $\mathbf{succ}_T(\tau) \in \mathcal{U}$ for every $\tau \in T$. In this case $T \subseteq \omega^{<\omega}$.

Lemma

If $\mathcal{U}$ is an ultrafilter then the following are equivalent:

1. Player 1 *has no winning strategy in the game* $\eth_s(\mathcal{U})$.
2. *Every $\mathcal{U}$-S-tree has a branch whose range is in $\mathcal{U}$.*
3. *For every $P : [\omega]^2 \to 2$ there is $A \in \mathcal{U}$ such that P is constant on $[A]^2$; in other words, $\omega \to (\mathcal{U})^2_2$*
4. *For every $k \in \omega$ and every $P : [\omega]^k \to 2$ there is $A \in \mathcal{U}$ such that P is constant on $[A]^k$; in other words, $\omega \to (\mathcal{U})^k_2$*
5. *For any $F : \omega \to \omega$ there is $A \in \mathcal{U}$ such that F is either constant or one-to-one on A.*
6. *For any family $\{A_n\}_{n\in\omega} \subseteq \mathcal{U}$ such that $A_{n+1} \subseteq A_n$ there is $A \in \mathcal{U}$ such that $|A \cap (A_n \setminus A_{n+1})| \leq 1$ for all n.*

An ultrafilter satisfying any of these properties is called **selective**.

Proof that 1 implies 3.

Let $P : [\omega]^2 \to 2$. Let T be the $\mathcal{U}$-tree consisting of all τ such that there are $Z_\tau \in \mathcal{U}$ and $f_\tau : |\tau| \to 2$ such that $P(n, m) = f_\tau(n)$ for all $n \in \text{domain}(\tau)$ and $m \in Z_\tau$. Let B be a branch of T whose range belongs to $\mathcal{U}$.

Note that $f = \bigcup_m f_{B \restriction m}$ is a function. Let $J \in 2$ be such that $f^{-1}\{J\} \in \mathcal{U}$ and note that $f^{-1}\{J\}$ is homogeneous.

$\square$

3 implies 4.

Proceed by induction on k starting with $k = 2$. Assume that for every $P : [\omega]^k \to 2$ there is $A \in \mathcal{U}$ such that P is constant on $[A]^k$ and let $Q : [\omega]^{k+1} \to 2$. Now define $Q^* : [\omega]^k \to 2$ by $Q^*(a) = i$ if and only if $Z_a = \{k \in \omega \setminus a \mid Q(a \cup \{k\}) = i\} \in \mathcal{U}$. Using the induction hypothesis it is possible to find $B \in \mathcal{U}$ and $J \in 2$ such that $Q^*(a) = J$ for all $a \in [B]^k$.

Now define $Q^{**} : [B]^2 \to 2$ by

$$Q^{**}(\{n, m\}) = \begin{cases} 0 & \text{if } n < m \ \& \ m \in \bigcap_{a \in [n]^{k-1}} Z_{a \cup \{n\}} \\ 1 & \text{otherwise.} \end{cases}$$

Now let $A \subseteq B$ be such that $A \in \mathcal{U}$ and Q^{**} is constant on $[A]^2$.

Continuation of 3 implies 4.

The first thing to notice is that it must be the case that $Q^{**}(a) = 0$ for all $a \in [A]^2$. To see this, let $n \in A$ and let $W = \bigcap_{a \in [n]^{k-1}} Z_{a \cup \{m\}} \in \mathcal{U}$. Then let $m \in A \cap W$ be such that $n < m$ and note that $\{n, m\} \in [A]^2$ and $Q^{**}(\{n, m\}) = 0$.

It now suffices to check that $Q(a) = J$ for all $a \in [A]^{k+1}$. To see this let $a = a_0 \cup \{m\} \cup \{n\}$ where $\max(a_0) < m < n$. Then $Q^{**}(\{n, m\}) = 0$ and so $n \in Z_{a_0 \cup \{m\}}$. Since $Q^*(a_0 \cup \{m\}) = J$ it follows that

$$Q(a) = Q((a_0 \cup \{m\}) \cup \{n\}) = J.$$

□

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Definition

Recall that $\mathfrak{d}$ is the least cardinality of a dominating family in ω^ω.

Theorem (Ketonen [8])

If $\mathfrak{d} = \mathfrak{c}$ then there is a P-point. Indeed, any filter with a base of cardinality less than $\mathfrak{c}$ can be extended to a P-point. (This stronger property is known as the generic existence of P-points)

The proof will use the following easy lemma at a crucial point.

Lemma

If $\mathcal{G}$ is a family of infinite **partial** *functions from ω to ω and $|\mathcal{G}| < \mathfrak{d}$ then there is $f : \omega \to \omega$ such that for each $g \in \mathcal{G}$ there are infinitely many n in the domain of g such that $f(n) > g(n)$.*

Proof of Theorem.

Let $\{F_\xi\}_{\xi \in \mathfrak{c}}$ enumerate ω^ω. Construct filters $\mathcal{F}_\xi$ such that:

1. if $\xi \in \eta$ then $\mathcal{F}_\xi \subseteq \mathcal{F}_\eta$
2. $\mathcal{F}_\eta$ has a base of cardinality $|\eta| \cdot \aleph_0$
3. if $F_\xi^{-1}\{n\} \in \mathcal{F}_\xi^*$ for all n then there is $A \in \mathcal{F}_{\xi+1}$ such that $F_\xi \upharpoonright A$ is finite-to-one.
4. if there is some n such that $F_\xi^{-1}\{n\} \notin \mathcal{F}_\xi^*$ then there is some n such that $F_\xi^{-1}\{n\} \in \mathcal{F}_{\xi+1}$.

If this can be done, then let $\mathcal{F} = \bigcup_{\xi \in \mathfrak{c}} \mathcal{F}_\xi$ and note that $\mathcal{F}$ is an ultrafilter by considering the F_ξ that are 2-valued and applying Induction Hypothesis 4.

Continuation.

To see that $\mathcal{F}$ is a P-point use Corollary 7 and let $F : \omega \to \omega$. Let $\xi \in \mathfrak{c}$ be such that $F_\xi = F$. If $F^{-1}\{n\} \in \mathcal{F}_\xi^*$ for all n then there is $A \in \mathcal{F}_{\xi+1} \subseteq \mathcal{F}$ such that $F \restriction A$ is finite-to-one. Otherwise there is n such that $F^{-1}\{n\} \in \mathcal{F}_{\xi+1} \subseteq \mathcal{F}$ and so F is constant on a set in $\mathcal{F}$.

To carry out the induction, let $\mathcal{F}_0 = \mathbb{F}^*$. At limit stages simply take unions, so suppose that $\mathcal{F}_\xi$ has been constructed. If there is some m such that $F_\xi^{-1}\{m\} \notin \mathcal{F}^*$ then let $\mathcal{F}_{\xi+1}$ be the filter generated by $\mathcal{F}_\xi \cup \{F_\xi^{-1}\{m\}\}$ and note that $F_\xi^{-1}\{m\} \in \mathcal{F}_{\xi+1}$.
Hence it will be assumed that $F_\xi^{-1}\{m\} \in \mathcal{F}^*$ for all m.
Let

$$X = \bigcup_{\{m \in \omega \mid |F_\xi^{-1}\{m\}| = \aleph_0\}} F_\xi^{-1}\{m\}.$$

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Continuation.

The first case to consider is $\omega \setminus X \in \mathcal{F}_\xi^+$. Then $\mathcal{F}_\xi \cup \{\omega \setminus X\}$ generates a filter consisting of infinite sets. Let $\mathcal{F}_{\xi+1}$ be this filter and note that $\omega \setminus X \in \mathcal{F}_{\xi+1}$ and $F_\xi \restriction \omega \setminus X$ is finite-to-one.
The other possibility is that $X \in \mathcal{F}_\xi$. Let

$$R = \left\{ m \in \omega \;\middle|\; |F_\xi^{-1}\{m\}| = \aleph_0 \right\}.$$

Note that if $Z \in \mathcal{F}_\xi$ then $Z \cap X \in \mathcal{F}_\xi$ and hence $Z \cap X \setminus F_\xi^{-1}(a) \neq \varnothing$ for each $a \in [R]^{<\aleph_0}$. Hence for each such Z there are infinitely many $r \in R$ such that $Z \cap F_\xi^{-1}\{r\} \neq \varnothing$. Let D_Z be any function whose domain is

$$\left\{ r \in R \;\middle|\; Z \cap F_\xi^{-1}\{r\} \neq \varnothing \right\}$$

such that $D_Z(r) \cap Z \cap F_\xi^{-1}\{r\} \neq \varnothing$ for each r in the domain of D_Z. Note that the domain of each D_Z is an infinite subset of R.

Continuation.

Let $\mathcal{B}$ be a base for $\mathcal{F}_\xi$ of cardinality less than $\mathfrak{c}$ and let $\mathcal{D} = \{D_Z \mid Z \in \mathcal{B}\}$. Up to this point no extra hypothesis has been needed, but it is now possible to use the easy lemma and the fact that $|\mathcal{B}| < \mathfrak{d}$ to find $f : \omega \to \omega$ such that for each $Z \in \mathcal{B}$ there are infinitely many r in the domain of D_Z such that $f(r) > D_Z(r)$. Let $W = \bigcup_{r \in R} F_\xi^{-1}\{r\} \cap f(r)$.

It is immediate that $F_\xi \restriction W$ is finite-to-one. Moreover, $W \cap Z$ is infinite for each $Z \in \mathcal{B}$ and hence $W \in \mathcal{F}_\xi^+$. Letting $\mathcal{F}_{\xi+1}$ be generated by $\mathcal{F}_\xi \cup \{W\}$ completes this inductive step. □

Theorem (Ketonen [8])

If $\mathbf{cov}(\mathcal{M}) = \mathfrak{c}$ *then there is a selective ultrafilter.*

Definition

If $\mathcal{U}$ *is an ultrafilter such that for any finite-to-one function* $F : \omega \to \omega$ *there is* $A \in \mathcal{U}$ *such that* F *is either constant or one-to-one on* A *is known as a* **Q-point**.

The following characterization of selective ultrafilters is immediate from Corollary 7 and Condition 5 of Lemma 2.

Lemma

An ultrafilter $\mathcal{U}$ *is selective if and only if it is both a P-point and a Q-point.*

If **cov**$(\mathcal{M}) = \mathfrak{c}$ then there is a Q-Point that is not a P-point and we will see later that there can P-points that are not Q-points. Miller showed [10] that there are no Q-points in Laver's model for the Borel Conjecture. The following remains open.

Question (Miller)

Is it consistent that there are no P-points **and** *no Q-points?*

More properties of P-points

Rosen [12] asked the natural question whether the condition $\omega \to (\mathcal{U})^2_2$ of the characterization of selective ultrafilters might be replaced by the assertion $\omega \to [\mathcal{U}]^k_m$ omitting a colour.

Theorem (Rosen)

Assuming $2^{\aleph_0} = \aleph_1$ for each $n \in \mathbb{N}$ there are ultrafilters $\mathcal{U}$ such that $\omega \to [\mathcal{U}]^2_{n+1}$ yet the relation $\omega \to [\mathcal{U}]^2_n$ fails.

Theorem (Blass, Dobrinen and Raghavan [3])

For each dimension n there is an integer $T(n)$ such that $\omega \to [\mathcal{U}]^n_{T(n)-1}$ holds if and only if $\mathcal{U}$ is a P-point and, moreover, there are non-P-points $\mathcal{U}$ such that $\omega \to [\mathcal{U}]^n_{T(n)}$ holds.

It should be mentioned that earlier characterizations of P-points along these lines were also known. However, the statements require some technicalities.

Definition (Baumgartner [1])

*Given $G \subseteq [X]^k$ the notation $\omega \to [\mathcal{U}, G]^k$ means the following: For any partition $P : [\omega]^k \to 2$ either there is $A \in \mathcal{U}$ such that $P(a) = 0$ for each $a \in [A]^k$ or there is a one-to-one mapping $h : X \to \omega$ such that $P(\{h[x]\}) = 1$ for each $x \in G$. The ultrafilter $\mathcal{U}$ is said to be an m-**arrow ultrafilter** if $m \geq 3$ and $\omega \to [\mathcal{U}, [m]^2]^2$ and $\mathcal{U}$ is said to be an **arrow ultrafilter** if it is a m-arrow ultrafilter for all m.*

In the case of $k = 2$ it may be more instructive to think of graphs. The relation $\omega \to [\mathcal{U}, G]^2$ holds if for every $P : [\omega]^2 \to 2$ there is either a set $A \in \mathcal{U}$ such that P restricted to $[A]^2$ is constantly 0 or there is an isomorphic copy of the graph G on some subset of ω whose edges are coloured 1 by P.

Theorem (Baumgartner and Taylor [2])

For an ultrafilter $\mathcal{U}$ the following are equivalent:

1. *$\mathcal{U}$ is selective*
2. $\omega \to [\mathcal{U}, [\omega]^2]^2$
3. $\omega \to [\mathcal{U}, [4]^3]^3$.

Kanamori showed that there is 3-arrow ultafilter if and only if there is P-point, even though the 3-arrow ultrafilters may not themselves be P-points.

Question

Is it consistent to have no $k+1$-arrow ultrafilters, but to have a k-arrow ultrafilter.

Under $\mathfrak{p} = \mathfrak{c}$ Baumgartner and Taylor show there are k-arrow P-points that are not $k+1$-arrow ultrafilters. Selective ultrafilters are arrow ultrafilters. One could still ask if for a given k and m it is consistent to have a P-point that is k-arrow and not $k+1$-arrow but no P-point that is m-arrow and not $m+1$-arrow.

Nešetřil [11] showed that $\omega \to [\mathcal{U}, F]^2$ for every countable, acyclic graph with finite components and every ultrafilter $\mathcal{U}$. However, assuming $\mathbf{cov}(\mathcal{M}) = \mathfrak{c}$, he showed there is a P-point $\mathcal{U}$ (which he called a Riga P-point) such that $\omega \to [\mathcal{U}, F]^2$ if and only if F is a countable, acyclic graph with finite components. In other words, under $\mathbf{cov}(\mathcal{M}) = \mathfrak{c}$, P-points cannot be characterized as the only ultrafilters satisfying $\omega \to [\mathcal{U}, G]^2$ for some graph G or even for some family of graphs.

Note that a consistency result along these lines is all that can be hoped for since it will be shown in the next lecture that it is consistent that there is a P-point but every P-point is selective. On the other hand, it has already been seen that $\omega \to [\mathcal{U}, K_{\aleph_0}]^2$ holds for any selective ultrafilter $\mathcal{U}$. Hence in this model the arrow relation $\omega \to [\mathcal{U}, K_{\aleph_0}]^2$ characterizes the P-points, although in a somewhat degenerate way.

Theorem (Nešetřil)

If $\mathbf{cov}(\mathcal{M}) = \mathfrak{c}$ *then there is a P-point* $\mathcal{U}$ *(that is not a Q-point) and has the following property: A countable graph* G *is acyclic with only finite components if and only if* $\omega \to [\mathcal{U}, G]^2$.

Proof of Theorem ▸ Jump to question.

The argument will use the following result of Erdös : For any integer k there is a finite graph G_k with vertices V_k such that:

- $\{V_k\}_{k\in\omega}$ partitions ω into finite sets
- the chromatic number of G_k is k
- G_k has no n-cycle for any $n \leq k$.

Let $G = \bigcup_k G_k = \oplus_k G_k$.

Continuation of Proof.

Now construct inductively filters $\mathcal{F}_\xi$ for $\xi \in \mathfrak{c}$ such that:

1. $\mathcal{F}_\xi \subseteq \mathcal{F}_\eta$ if $\xi \in \eta$
2. $\mathcal{F}_\xi$ is generated by a set of cardinality less than $\mathfrak{c}$
3. if $X \in \mathcal{F}_\xi$ and $m \in \omega$ then the chromatic number of $G \cap [X]^2$ is greater than m; in other words, there is some k such that the chromatic number of $G_k \cap [X]^2$ is greater than m.

Claim

Given $\mathcal{F}_\xi$ and $X \subseteq \omega$ there is $\mathcal{F}_{\xi+1}$ satisfying Inductive Condition 1 to 3 such that either $X \in \mathcal{F}_{\xi+1}$ or $X \in \mathcal{F}^*_{\xi+1}$.

Proof of claim.

The fact that it is possible to add either X or it complement to $\mathcal{F}_\xi$ while preserving Inductive Conditions 3 follows from the fact that if H is a graph with vertices W and $W = W_0 \cup W_1$ then the sum of the chromatic numbers of $H \cap [W_0]^2$ and $H \cap [W_1]^2$ is at least the chromatic number of H. □

Claim

Given $\mathcal{F}_\xi$ and a partition $\{A_n\}_{n\in\omega}$ of ω such that $A_n \in \mathcal{F}^*_\xi$ for each n there is $\mathcal{F}_{\xi+1}$ satisfying Inductive Conditions 1 to 3 and $X \in \mathcal{F}_{\xi+1}$ such that $|X \cap A_n| < \aleph_0$ for each $n \in \omega$.

Proof of claim.

Let $\mathbb{P} = \bigcup_{n\in\omega} \prod_{j\in n} [A_j]^{<\aleph_0}$ ordered by inclusion. This is a countable partial order and so the hypothesis $\mathbf{cov}(\mathcal{M}) = \mathfrak{c}$ can be applied to meet fewer than $\mathfrak{c}$ dense sets. Let $\mathcal{B} \subseteq \mathcal{F}_\xi$ be a set of cardinality less than $\mathfrak{c}$ generating $\mathcal{F}_\xi$. For each $X \in \mathcal{B}$ and $m \in \omega$ let $D(X, m)$ be the set of $c \in \mathbb{P}$ such that $G \cap [\bigcup_j c(j) \cap X]^2$ has chromatic number greater than m.

Continuation of proof of claim.

To see that each $D(X, m)$ is dense in $\mathbb{P}$ let $c \in \mathbb{P}$ with domain ℓ and note that that $X \setminus \bigcup_{i\in\ell} A_i \in \mathcal{F}_\xi$. By Induction Hypothesis 3 it follows that there is some $k \geq \ell$ such that the chromatic number of $G_k \cap [X \setminus \bigcup_{i\in\ell} A_i]^2$ is greater than m. Extend c to c' with domain m so that

$$\bigcup_{j=\ell}^{m-1} c'(j) \supseteq V_k \cap X \setminus \bigcup_{i\in\ell} A_i.$$

Hence it is possible to find a generic filter $\mathbb{G} \subseteq \mathbb{P}$ such that $\mathbb{G} \cap D(Z, m) \neq \varnothing$ for all $m \in \omega$ and $Z \in \mathcal{B}$. Let $X = \bigcup_{c\in\mathbb{G}} \bigcup_{j\in\omega} c(j)$ and note that letting $\mathcal{F}_{\xi+1}$ be generated by $\{X\} \cup \mathcal{F}_\xi^+$ will satisfy Induction Hypotheses 1 to 3. Moreover, $|X \cap A_n| < \aleph_0$ for all n.

□

Continuation of proof of theorem.

Using an appropriate enumeration of the instances of Claim 1 and Claim 2 it will follow that $\mathcal{F} = \bigcup_{\xi \in \mathfrak{c}} \mathcal{F}_\xi$ is a P-point. It has already been seen that $\omega \to [\mathcal{F}, F]^2$ for any countable, acyclic graph F with only finite components. To see that $\omega \not\to [\mathcal{F}, H]^2$ otherwise, let H be given and suppose that the vertices of H are ω.

There are two cases to consider. The first case is that H has an infinite component C. If P be the characteristic function of G then no set in $\mathcal{F}$ is homogeneous for P. If $e \in \omega^\omega$ is a one-to-one mapping it must be shown that there is an edge of H whose image under e is not an edge of G. Let k be such that $e^{-1}(V_k) \cap C \neq \varnothing$. Since C is infinite and $e^{-1}(V_k)$ is finite it follows that $e^{-1}(V_k)$ is not a component of H. Hence there is some $n \in C \setminus e^{-1}(V_k)$ and $m \in e^{-1}(V_k) \cap C$ such that $\{n, m\}$ is an edge of H. But then $e(n) \notin V_k$ and $e(m) \in V_k$ so $\{e(n), e(m)\}$ is not an edge of G.

Conclusion of proof of theorem.

The other possibility is that H contains a k-cycle $\{a_0, a_1, \ldots, a_k\}$ for some k. In this case let $P : [\omega]^2 \to 2$ be defined by $P(a) = 1$ if and only if there is some $j > k$ such that $a \in G_j$. If $e \in \omega^\omega$ is a one-to-one mapping such that $P(\{e(n), e(m)\}) = 1$ if $\{n, m\}$ is an edge of H then it follows that $P(\{e(a_i), e(a_{i+1})\}) = 1$ for each $i \in k$. Since $a_0 = a_k$ it follows that there is some $m > k$ such that $\{e(a_0), e(a_1), \ldots, e(a_k)\} \subseteq V_m$. But this contradicts that G_m contains no k-cycle. □

Question

It has already been noted that under the Continuum Hypothesis it is not possible to characterize P-points by a class of graphs such that $\mathcal{U}$ is a P-point if and only if $\omega \to [\mathcal{U}, G]^2$ for all graph G from the class. On the other hand, if there is a P-point, but all P-points are selective then $\mathcal{U}$ is a P-point if and only if $\omega \to [\mathcal{U}, K_{\aleph_0}]^2$.

Question

For which classes of graphs $\mathcal{G}$ is it consistent that $\mathcal{U}$ is a P-point if and only $\omega \to [\mathcal{U}, G]^2$ for all $G \in \mathcal{G}$.

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