# V. Dirac fields and their quantization

## Lorentz transformations revisited

Recall: Lorentz transform $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$
where $\Lambda^\mu{}_\lambda \Lambda^\nu{}_\kappa g^{\lambda\kappa} = g^{\mu\nu}$

Klein-Gordon field transforms:

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)$$
$$(\text{since } \phi'(x') = \phi(x))$$

Lorentz invariance of K-G theory:

$\phi'(x)$ satisfies KG equation if $\phi(x)$ does $\iff$ $\mathcal{L}$ is a Lorentz scalar.

KG eqn: $(\partial^2 + m^2)\phi(x) = 0$

$$\partial_\mu \phi(x) \to \partial_\mu \phi'(x) = \partial_\mu \phi(\Lambda^{-1}x)$$
$$= (\Lambda^{-1})^\nu{}_\mu \underbrace{(\partial_\nu \phi)(\Lambda^{-1}x)}_{\frac{\partial \phi(\Lambda^{-1}x)}{\partial(\Lambda^{-1}x)^\nu}}$$

$$\partial^2\phi(x) \to \partial^2\phi'(x) = g^{\mu\nu}(\Lambda^{-1})^\lambda{}_\mu (\Lambda^{-1})^\kappa{}_\nu (\partial_\lambda \phi)(\partial_\kappa \phi)$$
$$= (\partial^2\phi)(\Lambda^{-1}x)$$

$$(\partial^2 + m^2)\phi(x) \to (\partial^2 + m^2)\phi'(x) = (\partial^2 + m^2)\phi(\Lambda^{-1}x) = 0$$

Lagrangian:

$$\mathcal{L}(x) \longrightarrow \mathcal{L}'(x) = \frac{1}{2}(\partial_\mu \phi')^2 - \frac{1}{2}m^2 \phi'^2$$

$$= \frac{1}{2}(\Lambda^{-1})^\lambda{}_\mu (\Lambda^{-1})^\kappa{}_\nu \, g^{\mu\nu} (\partial_\lambda \phi)(\partial_\kappa \phi)$$

$$- \frac{1}{2}m^2 \phi^2 \qquad \leftarrow \text{evaluated at } \Lambda^{-1}x.$$

$$= \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 = \mathcal{L}(\Lambda^{-1}x)$$

$$S = \int d^4x \, \mathcal{L}(x) \longrightarrow S' = \int d^4x \, \mathcal{L}(\Lambda^{-1}x) = S \qquad \text{invariant}$$

Scalar field has the simplest transformation property; it is invariant (up to the argeument of the field).

More complicated example: vector field $A^\mu(x)$

$$A^\mu(x) \longrightarrow A'^\mu(x) = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x)$$

$$\partial^\mu A^\nu(x) \to \partial^\mu A'^\nu(x) = g^{\mu\lambda} \partial_\lambda \left(\Lambda^\nu{}_\kappa A^\kappa(\Lambda^{-1}x)\right)$$

$$= g^{\mu\lambda} \Lambda^\nu{}_\kappa (\Lambda^{-1})^\sigma{}_\lambda (\partial_\sigma A^\kappa)$$

$$= \Lambda^\mu{}_\tau \Lambda^\nu{}_\kappa \partial^\tau A^\kappa$$

using

$$g^{\mu\lambda}(\Lambda^{-1})^\sigma{}_\lambda = \Lambda^\mu{}_\tau \Lambda^\lambda{}_\alpha g^{\tau\alpha} (\Lambda^{-1})^\sigma{}_\lambda$$

$$= \Lambda^\mu{}_\tau g^{\tau\sigma}$$

So $F^{\mu\nu} \rightarrow F'^{\mu\nu} = \Lambda^{\mu}{}_{\lambda} \Lambda^{\nu}{}_{\kappa} F^{\lambda\kappa}$

$F_{\mu\nu} \rightarrow F'_{\mu\nu} = (\Lambda^{-1})^{\lambda}{}_{\mu} (\Lambda^{-1})^{\kappa}{}_{\nu} F_{\lambda\kappa}$

$$\mathcal{L} \rightarrow \mathcal{L}' = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} (\Lambda^{-1})^{\lambda}{}_{\mu} (\Lambda^{-1})^{\kappa}{}_{\nu} (\Lambda^{\mu}{}_{\alpha})(\Lambda^{\nu}{}_{\beta}) \cdot F^{\alpha\beta} F_{\lambda\kappa}$$

$$= -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} = \mathcal{L}$$

<u>Group theory</u>: a group $G$ is a set of objects $\{g_1, g_2, \ldots\}$ ~~satisfying the properties~~ with a group operation "$\cdot$"

(1) closure: if $g_1, g_2$ are in $G$, then $g_1 g_2$ is in $G$.

(2) associativity: $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$

(3) identity: the identity element "$1$" is in $G$ such that $1 \cdot g = g \cdot 1 = g$.

(4) inverse: $g^{-1}$ is in $G$, such that $g \cdot g^{-1} = g^{-1} \cdot g = 1$.

~~Set of Lorentz transformations (Lorentz group)~~
~~$\{g_1, g_2, \ldots\}$ = all possible $\Lambda$ matrices $\Lambda^{\mu}{}_{\nu}$ (i.e. $\Lambda_1, \Lambda_2, \ldots$)~~
~~operation = matrix multiplication.~~

Discrete groups: $(g_1, g_2, \ldots, g_n)$

Continuous groups (Lie groups): infinite # of $\{g_1, \ldots\}$ parametrized by continuous parameters

example: $G$ = Lorentz group

$\{g_1, \ldots\}$ = matrices $\Lambda^{\mu}{}_{\nu}$ $(\Lambda_1, \Lambda_2, \ldots)$

group operation = matrix multiplication

Lorentz invariance of theory requires $\mathcal{L}$ is a scalar under Lorentz transformations $\mathcal{L} \to \mathcal{L}' = \mathcal{L}$. (action is invariant.) This occurs as long as each upper index is contracted into one lower index.

general arbitrary tensor: $T^{\mu\nu\lambda\dots} \to \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} \Lambda^{\lambda}{}_{\gamma} \dots T^{\alpha\beta\gamma\dots}$

$$\mathcal{L} = T^{\mu\dots} T'_{\mu\dots} = g_{\mu\nu} T^{\mu\dots} T'^{\nu\dots}$$

$$\to g_{\mu\nu} \Lambda^{\mu}{}_{\alpha} T^{\alpha\dots} \Lambda^{\nu}{}_{\beta} T'^{\beta\dots} = g_{\alpha\beta} T^{\alpha\dots} T'^{\beta\dots}$$

$$= T^{\mu\dots} T'_{\mu\dots} = \mathcal{L}$$

$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 - \frac{\mu}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4$ invariant

$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ invariant

Lorentz group: set of Lorentz transformation matrices $\{\Lambda_1, \Lambda_2, \dots\}$

- continuous group (Lie group): parametrized by continuous boost $\vec{\beta}$ and rotation $\vec{\theta}$, $\Lambda(\vec{\beta}, \vec{\theta})$.
- group operation is matrix multiplication.

Representation of a group :

$M(\Lambda)$ is an $n \times n$ matrix satisfying the same group properties as $\Lambda$:

$$M(\Lambda_1) M(\Lambda_2) = M(\Lambda_1 \Lambda_2)$$

Different types of particles in QFT correspond to different representations of the Lorentz group.

General $n$-component field $\Phi$ transforms as $\Phi(x) \to \Phi'(x) = M(\Lambda)\,\Phi(\Lambda^{-1}x)$

example: $M(\Lambda) = 1$ (trivial representation)
scalar field $\phi$ (spin 0)

$M(\Lambda) = \Lambda^{\mu}{}_{\nu}$ (fundamental representation)
vector field $A^{\mu}$ (spin 1 i.e. photon)

To find a representation of a group, first consider the generators of the group.

example: rotations and angular momentum.

Rotation operator $R(\theta) = \exp(-i\,\theta^i J^i)$
where $\underline{J} = \underline{x} \times \underline{p} = -i\, \underline{x} \times \underline{\nabla}$
$J^i = +i\,\varepsilon^{ijk} x^j \partial^k = \frac{1}{2} \varepsilon^{ijk} J^{jk}$
is angular momentum. where
$J^{jk} = i(x^j \partial^k - x^k \partial^j)$

Infinitesimal rotations: $R(\theta) = 1 - i\theta^i J^i$

$J^i$ satisfies commutation relations

$$[J^i, J^j] = i\varepsilon^{ijk} J^k$$

This is called the Lie algebra of the group.
Find representation of group by finding $n\times n$ matrices that satisfy the Lie algebra and then exponentiating them.

e.g. $J^i \rightarrow \frac{\sigma^i}{2}$ spin-$\frac{1}{2}$ representation of rotations.

$$M(R) = \exp(-i\theta^i\sigma^i/2)$$

Generator of Lorentz transformations: (generalize $J^{ij}$)

$$J^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu)$$

commutator:

$$\begin{aligned}[J^{\mu\nu}, J^{\rho\sigma}] &= -[x^\mu\partial^\nu, x^\rho\partial^\sigma] \pm \text{perm.} \\ &= -x^\mu\partial^\sigma g^{\nu\rho} + x^\rho\partial^\nu g^{\sigma\mu} \pm \text{perm.} \\ &= i(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho})\end{aligned}$$

Guess: the form we want for $J^{\mu\nu}$ is

$$(J^{\mu\nu})_{\alpha\beta} = i(g^\mu{}_\alpha g^\nu{}_\beta - g^\mu{}_\beta g^\nu{}_\alpha)$$

check: LHS $= [J^{\mu\nu}, J^{\rho\sigma}]_{\alpha\beta} = (J^{\mu\nu})_\alpha{}^\gamma (J^{\rho\sigma})_{\gamma\beta} - (J^{\rho\sigma})_\alpha{}^\gamma (J^{\rho\sigma})_{\gamma\beta}$

$$\text{RHS} = i(g^{\nu\rho}(J^{\mu\sigma})_{\alpha\beta} + \ldots)$$

Infinitesimal Lorentz transformation: $\Lambda = 1 - \frac{i}{2}\omega_{\mu\nu} J^{\mu\nu}$

$$\Lambda^\alpha{}_\beta = g^\alpha{}_\beta - \frac{i}{2}\omega_{\mu\nu}(J^{\mu\nu})^\alpha{}_\beta$$

where $\omega_{\mu\nu}$ is infinitesimal parameter parametrizing boost & rotations. ($\omega_{\mu\nu} = -\omega_{\nu\mu}$ since $J^{\mu\nu}$ is antisymmetric)

e.g. Rotation about $\hat{z}$ axis by $\theta \ll 1$.

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c\theta & -s\theta & 0 \\ 0 & s\theta & c\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \simeq 1 - \theta \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now let $\omega_{12} = -\omega_{21} = \theta$ and other $\omega_{\mu\nu} = 0$.

$$\Lambda^\alpha{}_\beta = g^\alpha{}_\beta - \frac{i}{2}\omega_{12}(J^{12})^\alpha{}_\beta \times 2$$

$$= g^\alpha{}_\beta - i\theta \cdot i(g^{1\alpha} g^2{}_\beta - g^1{}_\beta g^{2\alpha})$$

$$= g^\alpha{}_\beta + \theta \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^\alpha{}_\beta$$

e.g. boosts about $\hat{z}$ axis by $\beta \ll 1$.

$$\Lambda = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \simeq \mathbb{1} + \beta \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Now let $\omega_{03} = -\omega_{30} = \beta$ and other $\omega_{\mu\nu} = 0$.

$$\Lambda^\alpha{}_\beta = g^\alpha{}_\beta - i\omega_{03} (J^{03})^\alpha{}_\beta$$

$$= g^\alpha{}_\beta - i\beta \cdot i \left( g^{0\alpha} g^3{}_\beta - g^0{}_\beta g^{3\alpha} \right)$$

$$= g^\alpha{}_\beta + \beta \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}^\alpha{}_\beta$$

So guess for $(J^{\mu\nu})_{\alpha\beta}$ corresponds to the usual representation of Lorentz transformations acting on 4-vectors.

## Dirac equation

We want to find the Lorentz representation corresponding to spin-1/2 particles. Analogous to $\sigma^i$ rep. for rotations in non-rel. QM.

Dirac trick: suppose we have a set of four $n \times n$ matrices $\gamma^\mu$ (Dirac matrices) satisfying

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \cdot \mathbb{1}_{n\times n}$$

Then $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ is an $n$-dim. $(n\times n)$ representation for $J^{\mu\nu}$. Then

$$M(\Lambda) = \exp\left(-\tfrac{i}{2}\omega_{\mu\nu} S^{\mu\nu}\right)$$

is a representation of the Lorentz group.

Consider 3-dim. Euclidean space: Dirac algebra becomes

$$\{\gamma^i, \gamma^j\} = -2\delta^{ij}\cdot\mathbb{1}$$

Can be satisfied for $\gamma^j = i\sigma^j$ (Pauli matrices)

$$S^{ij} = \frac{i}{4}[i\sigma^i, i\sigma^j] = -\frac{i}{4}[\sigma^i, \sigma^j] = \frac{1}{2}\varepsilon^{ijk}\sigma^k$$

(recall: we defined $J^i = \frac{1}{2}\varepsilon^{ijk}J^{jk}$
or $J^{ij} = \varepsilon^{ijk}J^k$)

so $S^k = \frac{1}{2}\sigma^k$ $\rightarrow$ usual spin-½ rep. for rotations

In 4-dim Minkowski spacetime, ~~[illegible]~~ the smallest set of 4 matrices obeying Dirac algebra is 4x4.

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$ chiral representation.

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$ where $\sigma^\mu = (1, \sigma^1, \sigma^2, \sigma^3) = (1, \underline{\sigma})$
$\bar{\sigma}^\mu = (1, -\sigma^1, -\sigma^2, -\sigma^3) = (1, -\underline{\sigma})$

(2x2 block form)

There are an infinite number of possible forms for these 4x4 Dirac matrices, but they are all equivalent (ie. related to each other by a unitary transformation $\gamma^\mu = U^\dagger \gamma'^\mu U$) and the chiral rep. is a useful form.

Boost generator: $S^{0i} = \frac{i}{4}[\gamma^0, \gamma^i] = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$

Rotation generator: $S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \frac{1}{2}\varepsilon^{ijk}\underbrace{\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}}_{=\Sigma^k}$

These matrices act on 4-component objects called Dirac spinors.
The Dirac spinor field is denoted $\psi(x) = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$

Rotation generator is just usual Pauli matrices acting on two-component spinors $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ and $\begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}$.
$\Sigma^k$ is the 4x4 analog of the spin operator $\sigma^k$

~~The Lorentz generators~~
Lorentz transform on $\psi(x)$:

$$\psi(x) \to \psi'(x) = M(\Lambda)\,\psi(\Lambda^{-1}x)$$
$$= \exp\left(-\tfrac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)\psi(\Lambda^{-1}x)$$

Since $S^{\mu\nu}$ is block diagonal, this 4x4 representation is reducible to two 2x2 reps. Write

$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ where $\psi_L = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, $\psi_R = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}$ are two-component spinors. (Weyl spinors)

Lorentz transforms act separately on $\psi_L$ and $\psi_R$:

$$\psi_L(x) \to \psi_L'(x) = \exp\left(-\frac{\omega^{0i}\sigma^i}{2} - \frac{i}{2}\omega^{ij}\varepsilon^{ijk}\sigma^k\right)\psi_L(\Lambda^{-1}x)$$
$$= \exp\left(-\tfrac{1}{2}\underline{\beta}\cdot\underline{\sigma} - \tfrac{i}{2}\underline{\theta}\cdot\underline{\sigma}\right)\psi_L(\Lambda^{-1}x)$$

where $\beta^i = \omega^{0i}$ and $\theta^i = \frac{1}{2}\varepsilon^{ijk}\omega^{jk}$

$$\psi_R(x) \to \exp(\tfrac{1}{2}\underline{\beta}\cdot\underline{\sigma} - \tfrac{i}{2}\underline{\theta}\cdot\underline{\sigma})\,\psi_R(\Lambda^{-1}x)$$

(note: boosts are not unitary transformations)

Dirac Lagrangian: want a Lorentz invariant Lagrangian.

note: $\psi^\dagger\psi$ is not Lorentz invariant because $M(\Lambda)$ is not unitary. $(S^{\mu\nu\dagger} \neq S^{\mu\nu})$

$$\psi^\dagger \to \psi^\dagger \exp(+\tfrac{i}{2}\omega_{\mu\nu} S^{\mu\nu\dagger})$$

consider $\bar{\psi} = \psi^\dagger\gamma^0$

note: $S^{ij\dagger} = S^{ij}$ (Hermitian) and $S^{ij}\gamma^0 = \gamma^0 S^{ij}$
$S^{0i\dagger} = -S^{0i}$ and $S^{0i}\gamma^0 = -\gamma^0 S^{0i}$
(and similar for $S^{i0\dagger}$).

so $(S^{\mu\nu})^\dagger\gamma^0 = \gamma^0 S^{\mu\nu}$

$$\begin{aligned}\bar{\psi}(x) \to \psi^\dagger(\Lambda^{-1}x)\exp(\tfrac{i}{2}\omega_{\mu\nu}S^{\mu\nu\dagger})\gamma^0 &= \psi^\dagger\gamma^0\exp(+\tfrac{i}{2}\omega_{\mu\nu}S^{\mu\nu}) \\ &= \bar{\psi}(\Lambda^{-1}x)\underbrace{\exp(\tfrac{i}{2}\omega_{\mu\nu}S^{\mu\nu})}_{M(\Lambda)^{-1} = M(\Lambda^{-1})}\end{aligned}$$

Thus $\bar{\psi}\psi$ is a Lorentz scalar.

Next, consider: $\bar{\psi}\gamma^\mu\partial_\mu\psi = g_{\mu\nu}\bar{\psi}\gamma^\mu\partial^\nu\psi$

$$g_{\mu\nu}\bar{\psi}\gamma^\mu\partial^\nu\psi \to \bar{\psi}M(\Lambda)^{-1}\gamma^\mu\Lambda^\nu{}_\lambda\partial^\lambda M(\Lambda)\psi\, g_{\mu\nu}$$

Consider $M^{-1}(\Lambda)\gamma^\mu M(\Lambda)$ :

First, evaluate:

$$[S^{\rho\sigma}, \gamma^\mu] = \frac{i}{4}[[\gamma^\rho, \gamma^\sigma], \gamma^\mu]$$

$$= \frac{i}{4}(\gamma^\rho\gamma^\sigma\gamma^\mu - \gamma^\mu\gamma^\rho\gamma^\sigma + \gamma^\mu\gamma^\sigma\gamma^\rho - \gamma^\sigma\gamma^\rho\gamma^\mu)$$

$$= \frac{i}{4}2(+\gamma^\rho g^{\sigma\mu} - \gamma^\sigma g^{\rho\mu} - \gamma^\sigma g^{\rho\mu} + g^{\sigma\mu}\gamma^\rho)$$

$$= \frac{i}{2}(g^\rho{}_\nu g^{\sigma\mu} - g^\sigma{}_\nu g^{\rho\mu} - g^\sigma{}_\nu g^{\rho\mu} + g^\rho{}_\nu g^{\sigma\mu})\gamma^\nu$$

$$= i(g^{\sigma\mu}g^\rho{}_\nu - g^\sigma{}_\nu g^{\rho\mu})\gamma^\nu = -(J^{\rho\sigma})^\mu{}_\nu \gamma^\nu$$

Consider infinitesimal transformation:

$$M^{-1}(\Lambda)\gamma^\mu M(\Lambda) = (1 + \tfrac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma})\gamma^\mu(1 - \tfrac{i}{2}\omega_{\lambda\kappa}S^{\lambda\kappa})$$

$$= \gamma^\mu + \tfrac{i}{2}\omega_{\rho\sigma}[S^{\rho\sigma}, \gamma^\mu] = \gamma^\mu - \tfrac{i}{2}\omega_{\rho\sigma}(J^{\rho\sigma})^\mu{}_\nu\gamma^\nu$$

For finite transform:

$$M^{-1}(\Lambda)\gamma^\mu M(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu$$

So $\gamma^\mu$ transforms as a real 4-vector.

Then $\bar\Psi\gamma^\mu\partial_\mu\Psi \to g_{\mu\nu}\Lambda^\mu{}_\kappa \Lambda^\nu{}_\lambda \bar\Psi\gamma^\kappa\partial^\lambda\Psi = \bar\Psi\gamma^\kappa\partial_\kappa\Psi$

is invariant.

The Dirac Lagrangian is:

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\psi$$

↑ $i$ needed to make $\mathcal{L}$ Hermitian.

Euler-Lagrange equation: write out spinor indices $a,b = 1,2,3,4$

$$\mathcal{L} = \bar{\Psi}_a (i\gamma^\mu_{ab}\partial_\mu - m\delta_{ab})\psi_b$$

Then $\dfrac{\partial \mathcal{L}}{\partial \bar{\Psi}_a} - \partial_\mu \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\Psi}_a)} = (i\gamma^\mu_{ab}\partial_\mu - m\delta_{ab})\psi_b = 0$

$$\Rightarrow \underline{(i\gamma^\mu\partial_\mu - m)\psi(x) = 0} \qquad \text{Dirac equation.}$$

Act on Dirac equation with $(-i\gamma^\nu\partial_\nu - m)$

$$\begin{aligned}
\Rightarrow \quad & (-i\gamma^\nu\partial_\nu - m)(i\gamma^\mu\partial_\mu - m)\psi(x) \\
&= (\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu + m^2)\psi(x) \\
&= (\tfrac{1}{2}\{\gamma^\nu,\gamma^\mu\}\partial_\nu\partial_\mu + m^2)\psi(x) \\
&= (\partial^2 + m^2)\psi(x)
\end{aligned}$$

So each spinor component $\psi_a(x)$ satisfies the KG eqn.
(this will give $p^2 = m^2$ as we want particles to satisfy)

Expanding $\psi = \begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}$, the Dirac eqn becomes:

$$(i\gamma^\mu\partial_\mu - m)\psi = \begin{pmatrix} -m & i\sigma^\mu\partial_\mu \\ i\bar{\sigma}^\mu\partial_\mu & -m \end{pmatrix}\begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix} = 0$$

$$\Rightarrow \quad i\bar{\sigma}^\mu\partial_\mu\psi_L = m\psi_R$$

$$i\sigma^\mu\partial_\mu\psi_R = m\psi_L$$

The mass term mixes $\psi_L$ and $\psi_R$. If $m=0$, then we have the ~~Weyl~~ Weyl equations:

$$i\bar{\sigma}^\mu \partial_\mu \psi_L = 0 \,, \quad i\sigma^\mu \partial_\mu \psi_R = 0$$

$\psi_L$ and $\psi_R$ decouple. For $m=0$, $\psi_L$ and $\psi_R$ correspond to fermions with left- and right-handed helicity.

So far, we haven't shown that the Dirac Lagrangian corresponds to particles with spin-½. This is shown only <u>after</u> we quantize and start talking about particles.

<u>Free particle (plane wave) solutions</u>

Klein-Gordon equation: $(\partial^2 + m^2)\psi(x) = 0$

$$\Rightarrow \quad \psi(x) = u(p)\, e^{-ip\cdot x} \qquad \text{plane wave solution.}$$

where $p^2 = m^2$ and $p^0 = E_p > 0$

$u(p)$ is a four-component spinor. Plug into Dirac eqn:

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = (\gamma^\mu p_\mu - m)\, u(p)\, e^{-ip\cdot x} = 0$$

$\Rightarrow$ $u(p)$ must satisfy $(\gamma^\mu p_\mu - m)\,u(p) = 0$.

Consider particle at rest: $p = (m, 0, 0, 0)$

Dirac eqn: $\underbrace{(\gamma^0 m - m\mathbb{1})}_{4\times4} u(m) = m \underbrace{\begin{pmatrix} -\mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix}}_{2\times2 \text{ blocks}} u(m) = 0$

$\rightarrow \quad u(m) = \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$ where $\xi$ is arbitrary two-component spinor.

Boost along $\hat{z}$-axis:

$$\begin{pmatrix} E_p \\ p^3 \end{pmatrix} = \exp\left(-\tfrac{i}{2}\omega_{03} J^{03}\cdot 2\right)\begin{pmatrix} m \\ 0 \end{pmatrix}$$

$$= \exp\left(-i\omega_{03}\, i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)\begin{pmatrix} m \\ 0 \end{pmatrix} = \exp\left(\omega_{03}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)\begin{pmatrix} m \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} m\cosh\omega_{03} \\ m\sinh\omega_{03} \end{pmatrix} = \begin{pmatrix} \gamma m \\ \beta\gamma m \end{pmatrix}$$

For infinitesimal boosts, $\omega_{03} \approx \beta$.
For finite boosts, $\omega_{03} = \tanh^{-1}\beta = \eta$ rapidity

Now boost $u(m)$:

$$u(p) = M(\Lambda)\, u(m) = \exp\left(-\tfrac{i}{2}\omega_{03} S^{03}\cdot 2\right) u(m)$$

$$= \exp\left(-i\omega_{03}\left(\tfrac{-i}{2}\right)\begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}\right) u(m)$$

$$= \exp\left(-\tfrac{\eta}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}\right) u(m)$$

$$= \left\{\cosh\left(\tfrac{\eta}{2}\right)\begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} - \sinh\left(\tfrac{\eta}{2}\right)\begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}\right\} u(m)$$

$$= \begin{pmatrix} e^{\eta/2}\left(\frac{1-\sigma^3}{2}\right) + e^{-\eta/2}\left(\frac{1+\sigma^3}{2}\right) & 0 \\ 0 & e^{\eta/2}\left(\frac{1+\sigma^3}{2}\right) + e^{-\eta/2}\left(\frac{1-\sigma^3}{2}\right) \end{pmatrix} u(m)$$

Note: $\cosh\eta = (e^{\eta} + e^{-\eta})/2 = \gamma$ $\Rightarrow$ $e^{\eta} = \gamma(1+\beta)$

$\sinh\eta = (e^{\eta} - e^{-\eta})/2 = \beta\gamma$ $\qquad e^{-\eta} = \gamma(1-\beta)$

$$\Rightarrow \quad e^{\eta/2} = \sqrt{\gamma + \gamma\beta} = \sqrt{\frac{E_p + p^3}{m}}$$

$$e^{-\eta/2} = \sqrt{\frac{E_p - p^3}{m}}$$

$$u(p) = \begin{pmatrix} \left[\sqrt{E_p + p^3}\left(\frac{1-\sigma^3}{2}\right) + \sqrt{E_p - p^3}\left(\frac{1+\sigma^3}{2}\right)\right]\xi \\ \left[\sqrt{E_p + p^3}\left(\frac{1+\sigma^3}{2}\right) + \sqrt{E_p - p^3}\left(\frac{1-\sigma^3}{2}\right)\right]\xi \end{pmatrix}$$

Can be expressed in simplified form as:

$$\boxed{u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix}}$$ valid for any $p^\mu$.

Note: $\left(\frac{1\pm\sigma^3}{2}\right)$ are projection operators.

$$\left(\frac{1\pm\sigma^3}{2}\right)^2 = \left(\frac{1\pm\sigma^3}{2}\right), \quad \mathbb{1} = \left(\frac{1+\sigma^3}{2}\right) + \left(\frac{1-\sigma^3}{2}\right)$$

$$\left(\frac{1+\sigma^3}{2}\right)\left(\frac{1-\sigma^3}{2}\right) = 0$$

e.g. $\sqrt{p\cdot\sigma} = \sqrt{p\cdot\sigma}\left[\left(\frac{1+\sigma^3}{2}\right) + \left(\frac{1-\sigma^3}{2}\right)\right]$

$$= \sqrt{E_p - p^3\sigma^3}\left(\frac{1+\sigma^3}{2}\right) + \sqrt{E_p - p^3\sigma^3}\left(\frac{1-\sigma^3}{2}\right)$$

$$= \sqrt{(E_p - p^3\sigma^3)\left(\frac{1+\sigma^3}{2}\right)^2} + \sqrt{(E_p - p^3\sigma^3)\left(\frac{1-\sigma^3}{2}\right)^2}$$

$$= \sqrt{E_p - p^3}\left(\frac{1+\sigma^3}{2}\right) + \sqrt{E_p + p^3}\left(\frac{1-\sigma^3}{2}\right)$$

Dirac equation: useful identity $(p\cdot\sigma)(p\cdot\bar\sigma) = m^2$

$$(p\cdot\sigma)(p\cdot\bar\sigma) = (E_p - \mathbf{p}\cdot\boldsymbol{\sigma})(E_p + \mathbf{p}\cdot\boldsymbol{\sigma}) = E_p^2 - p^i p^j \sigma^i \sigma^j$$
$$= E_p^2 - \tfrac{1}{2} p^i p^j \{\sigma^i, \sigma^j\} = E_p^2 - |\mathbf{p}|^2 = m^2$$

$$\gamma^\mu p_\mu\, u(p) = \begin{pmatrix} 0 & \sigma\cdot p \\ \bar\sigma\cdot p & 0 \end{pmatrix}\begin{pmatrix} \sqrt{\sigma\cdot p}\,\xi \\ \sqrt{\bar\sigma\cdot p}\,\xi \end{pmatrix} = \begin{pmatrix} \sqrt{\sigma\cdot p}\,\sqrt{(\sigma\cdot p)(\bar\sigma\cdot p)}\,\xi \\ \sqrt{\bar\sigma\cdot p}\,\sqrt{(\bar\sigma\cdot p)(\sigma\cdot p)}\,\xi \end{pmatrix}$$

$$= m\, u(p)$$ $u(p)$ satisfies Dirac eqn.

Spinors $\xi$ can be anything. Useful to take

$\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ i.e. spin up or down along $\hat{z}$ axis.

~~Consider limit of~~

Consider ultrarelativistic limit: $p^3 \gg m$

case: $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ right-handed particle

$$u(p) \cong \begin{pmatrix} \sqrt{E_p + p^3}\left(\frac{1-\sigma^3}{2}\right)\xi \\ \sqrt{E_p + p^3}\left(\frac{1+\sigma^3}{2}\right)\xi \end{pmatrix} \approx \sqrt{2E_p}\begin{pmatrix} 0 \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} \to \psi_R$$

case: $\xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ left-handed particle

$$u(p) \simeq \sqrt{2E_p}\begin{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 \end{pmatrix} \to \psi_L$$

In relativistic limit, $u(p)$ decomposes into two two-component spinors corresponding to helicity left and right.

Helicity operator: $h = \hat{p}\cdot\mathbf{S} = \frac{1}{2}\hat{p}^i \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$

$h = +\frac{1}{2}$ right-handed

$h = -\frac{1}{2}$ left-handed.

Alternative (negative frequency) plane wave solutions:

$$\psi(x) = v(p)\, e^{ip\cdot x}$$

where $p^0 = E_p = \sqrt{|\vec{p}|^2 + m^2} > 0$.
(sign of $p^0$ absorbed into exponential)

$v(p)$ satisfies $(\gamma^\mu p_\mu + m)\, v(p) = 0$

Solution: $v(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta \\ -\sqrt{p\cdot\bar\sigma}\,\eta \end{pmatrix}$

where $\eta$ are two-component spinors (not necessarily $\xi$)

Summary: each of $u(p)$ and $v(p)$ have two possible spinors
e.g. $\xi = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$ and $\eta = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$
$\Rightarrow$ Four spinors total.

$$u_s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_s \\ \sqrt{p\cdot\bar\sigma}\,\xi_s \end{pmatrix}, \quad s=1,2 \qquad \text{(spinors for particles)}$$

$$v_s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta_s \\ -\sqrt{p\cdot\bar\sigma}\,\eta_s \end{pmatrix}, \quad s=1,2 \qquad \text{(spinors for antiparticles)}$$

Dirac fermion has 4 degrees of freedom: particle vs. antiparticle (2) $\times$ spin up/down (2).

Useful relations:

$$u_s^\dagger(p)\, u_{s'}(p) = 2E_p\, \xi_s^\dagger \xi_{s'} = 2E_p\, \delta_{ss'}$$
$$v_s^\dagger(p)\, v_{s'}(p) = 2E_p\, \eta_s^\dagger \eta_{s'} = 2E_p\, \delta_{ss'}$$

if $\xi, \eta$ normalized to unity (assumed to be true)

$$\bar{u}_s(p)\, u_{s'}(p) = 2m\, \delta_{ss'} \qquad u^s(\vec{p})^\dagger v^{s'}(-\vec{p}) = 0$$
$$\bar{v}_s(p)\, v_{s'}(p) = -2m\, \delta_{ss'} \qquad v^s(\vec{p})^\dagger u^{s'}(\vec{p}) = 0$$

Spin sums:

$$\sum_s u_s(p)_a\, \bar{u}_s(p)_b = \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\, \xi_s \\ \sqrt{p\cdot\bar{\sigma}}\, \xi_s \end{pmatrix}_a \left( \xi_s^\dagger \sqrt{p\cdot\bar{\sigma}},\ \xi_s^\dagger \sqrt{p\cdot\sigma} \right)_b$$

4x4 matrix with components (a,b)

$$= \begin{pmatrix} \sqrt{p\cdot\sigma} \sum_s \xi_s \xi_s^\dagger \sqrt{p\cdot\bar{\sigma}} & \sqrt{p\cdot\sigma} \sum_s \xi_s \xi_s^\dagger \sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar{\sigma}} \sum_s \xi_s \xi_s^\dagger \sqrt{p\cdot\bar{\sigma}} & \sqrt{p\cdot\bar{\sigma}} \sum_s \xi_s \xi_s^\dagger \sqrt{p\cdot\sigma} \end{pmatrix}_{ab}$$

$$= \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar{\sigma} & m \end{pmatrix}_{ab} = (p\cdot\gamma + m)_{ab}$$

using $\sum_s \xi_s \xi_s^\dagger = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1\ 0) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}(0\ 1) = \mathbb{1}$

So $\sum_s u_s(p)\, \bar{u}_s(p) = \gamma\cdot p + m$

$$\sum_s v_s(p)\, \bar{v}_s(p) = \gamma\cdot p - m$$

# Dirac matrices and Dirac field bilinears

Bilinear = involves two Dirac fields (e.g. $\psi$ an $\bar{\psi}$)

Lorentz scalar : $\bar{\psi}\psi$

Lorentz vector : $\bar{\psi}\gamma^{\mu}\psi$

What are the most general structures $\bar{\psi}\Gamma\psi$ where $\Gamma$ is a $4\times4$ matrix formed from $\gamma$ matrices?

Any product of $\gamma$ matrices can be written as a sum of symmetric and antisymmetric terms:

$$\text{e.g.}\quad \gamma^{\mu}\gamma^{\nu} = \tfrac{1}{2}\{\gamma^{\mu},\gamma^{\nu}\} + \tfrac{1}{2}[\gamma^{\mu},\gamma^{\nu}]$$

$$= \gamma^{(\mu}\gamma^{\nu)} + \gamma^{[\mu}\gamma^{\nu]}$$

$$= g^{\mu\nu}\mathbb{1} + \gamma^{[\mu}\gamma^{\nu]}$$

Therefore only need to consider $\Gamma$ built from antisym. combinations of $\gamma$ matrices, since symmetric combinations can be reduced to $g^{\mu\nu}\times$(fewer $\gamma$ matrices).

There are sixteen possibilities:

$\Gamma = \mathbb{1}$ (1)

$= \gamma^{\mu}$ (4)

$\gamma^{[\mu}\gamma^{\nu]}$ (6)

$\gamma^{[\mu}\gamma^{\nu}\gamma^{\lambda]}$ (4)

$\gamma^{[\mu}\gamma^{\nu}\gamma^{\lambda}\gamma^{\kappa]}$ (1)

Introduce new matrix:

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\varepsilon_{\mu\nu\lambda\kappa}\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\kappa = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

$$(\varepsilon_{0123} = -1)$$

Satisfies:
$(\gamma^5)^\dagger = \gamma^5$
$\{\gamma^5, \gamma^\mu\} = 0$
$(\gamma^5)^2 = \mathbb{1}$

So $\gamma^{[\mu}\gamma^\nu\gamma^\lambda\gamma^{\kappa]} = -i\varepsilon^{\mu\nu\lambda\kappa}\gamma^5$

$$\gamma^{[\mu}\gamma^\nu\gamma^{\lambda]} = i\varepsilon^{\mu\nu\lambda\kappa}\gamma_\kappa\gamma^5$$

check: $\gamma^{[0}\gamma^1\gamma^{2]} = \gamma^0\gamma^1\gamma^2 = i\varepsilon^{0123}\gamma_3\gamma^5$
$= i\gamma_3\, i\gamma^0\gamma^1\gamma^2\gamma^3 = +\gamma^0\gamma^1\gamma^2$

So (factoring out the $\varepsilon$ tensors), the 16 possible structures are:

| | | |
|---|---|---|
| $\mathbb{1}$ | scalar (1) | $\bar\psi\psi$ |
| $\gamma^\mu$ | vector (4) | $\bar\psi\gamma^\mu\psi$ |
| $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu]$ | tensor (6) | $\bar\psi\sigma^{\mu\nu}\psi$ |
| $\gamma^\mu\gamma^5$ | axial vector (4) | $\bar\psi\gamma^\mu\gamma^5\psi$ |
| $\gamma^5$ | pseudoscalar (1) | $\bar\psi\gamma^5\psi$ |

These terms have well-defined Lorentz transformation properties.

$\bar\psi\psi \to \bar\psi\psi$
$\bar\psi\gamma^\mu\psi \to \Lambda^\mu{}_\nu\,\bar\psi\gamma^\nu\psi$
$\bar\psi\sigma^{\mu\nu}\psi \to \Lambda^\mu{}_\lambda\Lambda^\nu{}_\kappa\,\bar\psi\sigma^{\lambda\kappa}\psi$
$\bar\psi\gamma^\mu\gamma^5\psi \to \Lambda^\mu{}_\nu\,\bar\psi\gamma^\nu\gamma^5\psi$
$\bar\psi\gamma^5\psi \to \bar\psi\gamma^5\psi$

} has opposite transform under Parity compared to $\bar\psi\psi$ and $\bar\psi\gamma^\mu\psi$

These are the building blocks of Lagrangians.

$j^\mu = \bar\Psi \gamma^\mu \psi$ is vector current

$j^\mu_5 = \bar\Psi \gamma^\mu \gamma^5 \psi$ is axial-vector current

Note: Dirac equation is $i \partial_\mu \gamma^\mu \psi = m \psi$

$$(i \partial_\mu \gamma^\mu \psi)^\dagger \gamma^0 = -i(\partial_\mu \psi^\dagger)\gamma^{\mu\dagger}\gamma^0$$

$$= -i(\partial_\mu \psi^\dagger)\gamma^0 \underbrace{\gamma^0 \gamma^{\mu\dagger}\gamma^0}_{=\gamma^\mu}$$

$$= -i(\partial_\mu \bar\Psi)\gamma^\mu = m\bar\Psi$$

$$\partial_\mu j^\mu = \partial_\mu(\bar\Psi \gamma^\mu \psi) = (\partial_\mu \bar\Psi)\gamma^\mu \psi + \bar\Psi \gamma^\mu (\partial_\mu \psi)$$

$$= im\bar\Psi\psi + (-im)\bar\Psi\psi = 0$$

Vector current conserved by Dirac eqn.

$$\partial_\mu j^\mu_5 = 2im\bar\Psi\gamma^5\psi \neq 0$$

axial-vector current only conserved if $m=0$.

Chirality: $\gamma^5$ can be used to project out $\psi_L, \psi_R$.

$$P_L = \frac{1-\gamma^5}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad P_R = \frac{1+\gamma^5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$P_L \psi = P_L \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix}$$

$$P_R \psi = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix}$$

$\psi_{L,R}$ are called chiral fields.
In relativistic limit ($m \to 0$), chirality = helicity.

If $m = 0$, useful to define chiral currents:

$$j^\mu_L = \bar{\psi}\gamma^\mu P_L \psi, \quad j^\mu_R = \bar{\psi}\gamma^\mu P_R \psi$$

and $\partial_\mu j^\mu_{L,R} = 0$

Noether's theorem:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

$\alpha\Delta\psi = -i\alpha\psi$

$\mathcal{L}$ invariant under $\psi \to e^{-i\alpha}\psi \cong (1 - i\alpha)\psi$
$\bar{\psi} \to \bar{\psi}e^{+i\alpha} \cong \underbrace{\bar{\psi}(1+i\alpha)}_{\text{infinitesimal } \alpha}$

conserved current:

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_a)}\Delta\psi_a = i\bar{\psi}\gamma^\mu(-i)\psi = \bar{\psi}\gamma^\mu\psi$$

For $m=0$, $\mathcal{L}$ invariant under $\psi \to e^{-i\alpha\gamma^5}\psi$.
This is a chiral transformation: $\psi_L \to e^{i\alpha}\psi_L$
$\psi_R \to e^{-i\alpha}\psi_R$

Conserved current is $j^\mu_5$.

## Quantization of the Dirac field

$$\mathcal{L} = \bar{\psi}(i\gamma^0 \partial_0 + i\gamma^i \partial_i - m)\psi$$

canonical momentum! $\pi_a(\underline{x}) = \dfrac{\partial \mathcal{L}}{\partial \dot{\psi}_a} = i(\bar{\psi}\gamma^0)_a = i\psi_a^\dagger$

Hamiltonian:

$$\mathcal{H} = \pi_a \dot{\psi}_a - \mathcal{L} = -i\bar{\psi}\gamma^i\partial_i\psi + m\bar{\psi}\psi$$

$$H = \int d^3x \, \mathcal{H}$$

<u>Idea</u>: follow quantization prescription from KG theory
(1) postulate canonical commutation relations
(2) expand field in terms of creation/annihilation ops.
(3) diagonal $H$.

(1) canonical commutation relations: (Schrodinger fields)

$$[\psi_a(\underline{x}), \pi_b(\underline{x}')] = i\delta^3(\underline{x}-\underline{x}')\delta_{ab}$$

and $[\psi,\psi] = [\pi,\pi] = 0$.

$$[\psi_a(\underline{x}), \psi_b^\dagger(\underline{x}')] = \delta^3(\underline{x}-\underline{x}')\delta_{ab}$$

(2) expand $\psi$ in terms of plane wave solutions

$$\psi_a(\underline{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s \left( u_a^s(p)\, a_p^s\, e^{+ip\cdot x} + v_a^s(p)\, b_p^s\, e^{-ip\cdot x}\right)$$

where $[a_p^s, a_{p'}^{s'\dagger}] = [b_p^s, b_{p'}^{s'\dagger}] = (2\pi)^3\delta^3(p-p')\delta_{ss'}$

Check: $[\psi_a(\underline{x}), \psi_b^\dagger(\underline{x}')]$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{p'}}} \sum_{s,s'}$$

$$\times \left( u_a^s(p)\, u_b^{s'}(p')^\dagger\, e^{i(\underline{p}\cdot\underline{x} - \underline{p}'\cdot\underline{x}')} [a_{\underline{p}}^s, a_{\underline{p}'}^{s'\dagger}] \right.$$

$$\left. + v_a^s(+p)\, v_b^{s'}(+p')^\dagger\, e^{-i(\underline{p}\cdot\underline{x} - \underline{p}'\cdot\underline{x}')} [b_{+\underline{p}}^s, b_{+\underline{p}'}^{s'\dagger}] \right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_s \left( (uu^\dagger)_{ab}\, e^{i\underline{p}\cdot(\underline{x}-\underline{x}')} + (vv^\dagger)_{ab}\, e^{-i\underline{p}\cdot(\underline{x}-\underline{x}')} \right)$$

use spin sum:

$$\sum_s (uu^\dagger)_{ab} = \sum_s u_a^s(p)\, \bar{u}_b^s(p)\, \gamma^0 = \underbrace{(\gamma\cdot p + m)\gamma^0}^{(\gamma^0 E_p - \underline{\gamma}\cdot\underline{p} + m)\gamma^0}$$

$$\sum_s (vv^\dagger)_{ab} = \sum_s v_a^s(+p)\, \bar{v}_b^s(+p)\gamma^0 = (\gamma^0 E_p + \underline{\gamma}\cdot\underline{p} - m)\gamma^0$$

$$[\psi_a(\underline{x}), \psi_b^\dagger(\underline{x}')] = \int \frac{d^3p}{(2\pi)^3} e^{i\underline{p}\cdot(\underline{x}-\underline{x}')} \delta_{ab} = \delta^3(\underline{x}-\underline{x}')\,\delta_{ab}$$

note: $\psi \sim (a+b)$ and not $(a+b^\dagger)$ here
to preserve $[b, b^\dagger] = (2\pi)^3\delta^3(..)$ and not
have $[b, b^\dagger] = -(2\pi)^3\delta^3(..)$. Must have (+)
sign here to keep interpretation of $b^\dagger b$ as number
operator.

Evaluate Hamiltonian:

$$H = \int d^3x\, \bar{\Psi} \left( -i \underline{\gamma} \cdot \underline{\nabla} + m \right) \Psi$$

$$= \int d^3x \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{p'}}} \sum_{s,s'} \left( u^{s'}(p')^{\dagger} a_{p'}^{s'\dagger} e^{-ip'\cdot x} + v^{s'}(+p')^{\dagger} b_{p'}^{s'\dagger} e^{ip'\cdot x} \right)$$

$$\times \gamma^0 \left( -i \underline{\gamma} \cdot \underline{\nabla} + m \right) \left( u^s(p)\, a_p^s\, e^{ip\cdot x} + v^s(+p)\, b_p^s\, e^{-ip\cdot x} \right)$$

Note:

$$\left( -i \underline{\gamma} \cdot \underline{\nabla} + m \right) u^s(p)\, e^{ip\cdot x} = \left( p \cdot \underline{\gamma} + m \right) u^s(p)\, e^{ip\cdot x}$$
$$= \gamma^0 E_p\, u^s(p)\, e^{ip\cdot x}$$

$$\left( -i \underline{\gamma} \cdot \underline{\nabla} + m \right) v^s(+p)\, e^{-ip\cdot x} = \left( - p \cdot \underline{\gamma} + m \right) v^s(+p)\, e^{-ip\cdot x}$$
$$= \left( -\gamma^0 E_p \right) v^s(+p)\, e^{-ip\cdot x}$$

$$H = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_{s,s'} \Bigg( \underbrace{u^{s'}(p)^{\dagger} u^s(p)}_{2E_p \delta_{ss'}} a^{\dagger} a \cdot E_p - E_p \underbrace{v^{s'}(p)^{\dagger} v^s(p)}_{2E_p \delta_{ss'}} b^{\dagger} b$$
$$+ \underbrace{u^{s'}(p)^{\dagger} v^s(-p)}_{0} a^{\dagger} b\, (-E_p) + \underbrace{v^{s'}(-p)^{\dagger} u^s(p)}_{0} b^{\dagger} a\, (E_p) \Bigg)$$

$$= \int \frac{d^3p}{(2\pi)^3} \sum_s E_p \left( a_p^{s\dagger} a_p^s - b_p^{s\dagger} b_p^s \right)$$

This is bad. Energy is unbounded from below. Can lower energy arbitrarily by creating more and more particles with $b^{\dagger}$.

<u>Fix</u>: postulate of canonical commutation relations was incorrect. Require canonical <u>anti</u>commutation relation:

$$\{\psi_a(\underline{x}), \pi_b(\underline{x}')\} = i\delta^3(\underline{x}-\underline{x}')\delta_{ab}$$

$$\{\psi_a(\underline{x}), \psi_b^\dagger(\underline{x}')\} = \delta^3(\underline{x}-\underline{x}')\delta_{ab}$$

and $\{\psi_a(\underline{x}), \psi_b(\underline{x}')\} = \{\pi_a(\underline{x}), \pi_b(\underline{x}')\} = 0.$

Expand $\psi(\underline{x})$ as before:

$$\psi(\underline{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s\left(u^s(p)\,a_{\underline{p}}^s\,e^{+i\underline{p}\cdot\underline{x}} + v^s(p)\,b_p^s\,e^{-i\underline{p}\cdot\underline{x}}\right)$$

but now $a, b$ satisfy anticommutation relations.

$$\{a_{\underline{p}}^s, a_{\underline{p}'}^{s'\dagger}\} = \{b_{\underline{p}}^s, b_{\underline{p}'}^{s'\dagger}\} = (2\pi)^3\delta^3(\underline{p}-\underline{p}')\delta_{ss'}$$

Get same $H$ as before:

$$H = \int\frac{d^3p}{(2\pi)^3}E_p\sum_s\left(a_{\underline{p}}^{s\dagger}a_{\underline{p}}^s - b_{\underline{p}}^{s\dagger}b_{\underline{p}}^s\right)$$

But note: anticommutation relation for $\{b, b^\dagger\}$ is symmetric under $b \leftrightarrow b^\dagger$. So we can redefine $b \to b^\dagger$, $b^\dagger \to b$.

$$H = \int\frac{d^3p}{(2\pi)^3}E_p\sum_s\left(a_{\underline{p}}^{s\dagger}a_{\underline{p}}^s - b_{\underline{p}}^s b_{\underline{p}}^{s\dagger}\right)$$

$$= \int\frac{d^3p}{(2\pi)^3}E_p\sum_s\left(a_{\underline{p}}^{s\dagger}a_{\underline{p}}^s + b_{\underline{p}}^{s\dagger}b_{\underline{p}}^s\right) - \text{const.}$$

infinite const $= \displaystyle\int\frac{d^3p}{(2\pi)^3}E_p\cdot 2\,\underbrace{(2\pi)^3\delta^3(0)}_{V} = \overset{\uparrow}{\underset{\#\text{ of d.o.f.}}{4}}\times\left(\begin{array}{l}\text{infinite energy}\\ \text{for real scalar}\\ \text{field}\end{array}\right)$

Fermion states: $|\underline{p}, s\rangle = \sqrt{2E_p}\, a_{\underline{p}}^{s\dagger} |0\rangle$ 1 particle state

$|\widetilde{\underline{p}, s}\rangle = \sqrt{2E_p}\, b_{\underline{p}}^{s\dagger} |0\rangle$ 1 antiparticle state.

Note: $|\underline{p}, s; \underline{p}', s'\rangle = \sqrt{2E_p}\sqrt{2E_{p'}}\, a_{\underline{p}}^{s\dagger} a_{\underline{p}'}^{s'\dagger} |0\rangle$

two particle state:

$= -\sqrt{2E_p}\sqrt{2E_{p'}}\, a_{\underline{p}'}^{s'\dagger} a_{\underline{p}}^{s\dagger} |0\rangle$

$= -|\underline{p}', s'; \underline{p}, s\rangle$

States obey <u>Fermi-Dirac statistics</u>.
Antisymmetric under exchange of two ~~states~~ particles.

Pauli exclusion principle: two particles can't occupy same state.

$$|\underline{p}, s; \underline{p}, s\rangle = 2E_p \left(a_{\underline{p}}^{s\dagger}\right)^2 |0\rangle = 0$$

$$\text{since } \{a_{\underline{p}}^{s\dagger}, a_{\underline{p}}^{s\dagger}\} = 2(a_{\underline{p}}^{s\dagger})^2 = 0.$$

In deriving H, we had to swap $b \leftrightarrow b^\dagger$. What does this mean?
Consider a theory with $b, b^\dagger$ and no spin/momentum labels.

We have vacuum state: $|0\rangle$ such that $b|0\rangle = 0$
and excited (one-particle) state $|1\rangle = b^\dagger|0\rangle$.
Note: $|0\rangle$ and $|1\rangle$ are the only two states since $b^\dagger|1\rangle = b^\dagger b^\dagger|0\rangle = 0$

Swapping $b \leftrightarrow b^\dagger$ means swapping $|1\rangle \leftrightarrow |0\rangle$, and same rules apply: $b|0\rangle = 0$, $b^\dagger|0\rangle = |1\rangle$.

Pick which convention to use such that $|1\rangle$ has higher energy than $|0\rangle$ $(E_1 > E_0)$.

$\Rightarrow$ $b_p^{s\dagger}$ creates antiparticles with positive energy $E_p$.

Other convention: "vacuum" state $|0\rangle$ has <u>all</u> antiparticle states occupied, and $b_p^{s\dagger}|0\rangle$ ~~annihilates~~ removes an antiparticle of energy $E_p$ to create a <u>hole</u> of energy $-E_p$.

~~Quantized Dirac fields:~~

<u>Quantized (free) Dirac field:</u> Heisenberg picture

$$\psi(x) = e^{iHt}\,\psi(\underline{x})\,e^{-iHt}$$

Similar to KG fields, we get:

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left( u_s(p)\, a_p^s\, e^{-ip\cdot x} + v_s(p)\, b_p^{s\dagger}\, e^{+ip\cdot x} \right)$$

$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left( \bar{u}_s(p)\, a_p^{s\dagger}\, e^{ip\cdot x} + \bar{v}_s(p)\, b_p^s\, e^{-ip\cdot x} \right)$$

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s E_p \left( a_p^{s\dagger} a_p^s + b_p^{s\dagger} b_p^s \right)$$

$$\underline{P} = \int d^3x\, \psi^\dagger(-i\underline{\nabla})\psi = \int \frac{d^3p}{(2\pi)^3}\, \underline{p}\left( a_p^{s\dagger} a_p^s + b_p^{s\dagger} b_p^s \right)$$

$$= \int d^3x\, T^{0i}$$

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_a)} \partial_\nu \psi_a - \mathcal{L}\, g^\mu{}_\nu$$

$$= i\bar{\psi}\gamma^\mu \partial_\nu \psi - \mathcal{L}\, g^\mu{}_\nu$$

conserved charge: $Q = \int d^3x \, j^0 = \int d^3x \, \bar{\psi} \gamma^0 \psi = \int d^3x \, \psi^\dagger \psi$

$$= \int \frac{d^3p}{(2\pi)^3} \sum_s \left( a_p^{s\dagger} a_p^s + b_p^s b_p^{s\dagger} \right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \sum_s \left( a_p^{s\dagger} a_p^s - b_p^{s\dagger} b_p^s \right)$$

$$+ \text{(infinite constant)}$$

So $a_p^{s\dagger}$ creates particle with charge $+1$
$b_p^{s\dagger}$ creates particle with charge $-1$ $\Big\}$ opposite charges for particle/antiparticle.

## Spin of Dirac particles (expect $s = 1/2$)

Use Nother's theorem: angular momentum is conserved charge for rotations

Rotate $\psi(x)$ about $\hat{z}$ axis by $\theta \ll 1$

$$\psi(x) \to \psi'(x) = M(\Lambda)\,\psi(\Lambda^{-1}x)$$

where
$$M(\Lambda) = \mathbb{1} - \tfrac{i}{2}\theta\Sigma^3$$
$$\Lambda^{-1}x = (t,\ x+\theta y,\ y-\theta x,\ z)$$

$$\psi(x) \to \psi'(x) = \left(\mathbb{1} - \tfrac{i}{2}\theta\Sigma^3\right)\left(\psi(x) + \theta y\,\partial_x\psi - \theta x\,\partial_y\psi\right)$$

$$= \psi(x) + \theta\,\underbrace{\Delta\psi(x)}_{=\left(-\varepsilon^{ij3}x^i\partial_j - \frac{i}{2}\Sigma^3\right)\psi}$$

Recall conserved charge:

$$Q = \int d^3x\, j^0 = \int d^3x\, \frac{\partial\mathcal{L}}{\partial(\partial_0\psi_a)}\,\Delta\psi_a$$

Here $Q \to J^3$

$$J^3 = \int d^3x\, \bar{\psi}\, i\gamma^0\left(-\varepsilon^{ij3}x^i\partial_j - \tfrac{i}{2}\Sigma^3\right)\psi$$

Similar conserved charge for rotations about $\hat{x}, \hat{y}$ axes
$\to J^1, J^2$

$$\underline{J} = \int d^3x\, \psi^\dagger\Big(\underbrace{\underline{x}\times(-i\underline{\nabla})}_{\substack{\underline{L}\\ \text{orbital}}} + \underbrace{\tfrac{1}{2}\underline{\Sigma}}_{\substack{\underline{S}\\ \text{spin}}}\Big)\psi$$

Consider 1-particle state with spin $s$ and $\underline{p}=0$.

$$|\underline{p}=0, s\rangle = \sqrt{2m}\, a_0^{s\dagger} |0\rangle$$

Show $\quad J^3 |\underline{p}=0, s\rangle = \pm\frac{1}{2} |\underline{p}=0, s\rangle$

Note: $J^3 a_0^{s\dagger}|0\rangle = [J^3, a_0^{s\dagger}]|0\rangle \qquad$ since $J^3|0\rangle = 0$.

$$J^3 = \int d^3x \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}} \int \frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{p'}}} \sum_{r,r'} \left(u^{r'}(p')\, a_{p'}^{r'\dagger} e^{ip'\cdot x} + v^{r'}(p')\, b_{p'}^{r'} e^{-ip'\cdot x}\right)$$

$$\times \left( \underset{\checkmark}{\underline{x}\times(-i\nabla)} + \tfrac{1}{2}\underline{\Sigma}\right)^3 \left(u^r(p) a_p^r e^{-ip\cdot x} + v^r(p) b_p^{r\dagger} e^{ip\cdot x}\right)$$

can neglect orbital term in $[J^3, a_0^{s\dagger}]$

only $a_{p'}^{\dagger} a_p$ term in $[J^3, a_0^{s\dagger}]$ doesn't vanish.

$$[J^3, a_0^{s\dagger}]|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\sum_{r,r'} u^{r'}(p)^{\dagger}\, \tfrac{1}{2}\Sigma^3 u^r(p)\, [a_p^{r'\dagger} a_p^r, a_0^{s\dagger}]|0\rangle$$

$$[a_p^{r'\dagger} a_p^r, a_0^{s\dagger}]|0\rangle = \left(a_p^{r'\dagger} a_p^r a_0^{s\dagger} - a_0^{s\dagger} a_p^{r'\dagger} a_p^r\right)|0\rangle$$

$$= a_p^{r'\dagger} (2\pi)^3 \delta^3(p)\, \delta_{rs}$$

$$[J^3, a_0^{s\dagger}]|0\rangle = \frac{1}{2m}\sum_{r'} u^{r'}(0)^{\dagger}\, \tfrac{1}{2}\Sigma^3 u^s(0)\, a_0^{r'\dagger}$$

$$u^s(0) = \sqrt{m}\begin{pmatrix}\xi_s\\ \xi_s\end{pmatrix} \qquad u^{r'}(0)^{\dagger} = \sqrt{m}\left(\xi_{r'}^{\dagger}, \xi_{r'}^{\dagger}\right)$$

$$= \frac{1}{2}\underbrace{\left(\sum_{r'} \xi_{r'}^{\dagger} \sigma^3 \xi_s\right)}_{=\pm 1} a_0^{r'\dagger} = \pm\frac{1}{2} a_0^{s\dagger}$$

$\sigma^3\xi_1 = 1, \quad \sigma^3\xi_2 = -1$

So ~~[illegible]~~ for a particle state at rest:

$$J^3 |p=0, s=1\rangle = +\tfrac{1}{2} |p=0, s=1\rangle$$

$$J^3 |p=0, s=2\rangle = -\tfrac{1}{2} |p=0, s=2\rangle$$

where $\xi_1 = \begin{pmatrix}1\\0\end{pmatrix}$ spin up along $\hat{z}$ axis
$\xi_2 = \begin{pmatrix}0\\1\end{pmatrix}$ spin down along $\hat{z}$ axis

Next, consider 1-antiparticle state: $|p=0, s\rangle = \sqrt{2m}\, b_0^{s\dagger} |0\rangle$

Similarly, compute $[J^3, b_0^{s\dagger}] \sim [\underbrace{b_{p'}^{r'} b_p^{r\dagger}}, b_0^{s\dagger}]$

extra minus sign
$bb^\dagger = -b^\dagger b$.

For antiparticle states:

$$J^3 |p=0, s=1\rangle = -\tfrac{1}{2} |p=0, s=1\rangle$$
$$J^3 |p=0, s=2\rangle = +\tfrac{1}{2} |p=0, s=1\rangle$$

So $\eta_1 = \begin{pmatrix}1\\0\end{pmatrix}$ is spin-<u>down</u>
$\eta_2 = \begin{pmatrix}0\\1\end{pmatrix}$ is spin-<u>up</u> $\Big\}$ along $\hat{z}$ axis.

## Discrete symmetries

Parity (P)
Time-reversal (T)
Charge conjugation (C)

How does $\psi(x)$ transform under C, P, T ?

### Parity

$\underline{p} \xrightarrow{P} -\underline{p}$ reverse 3-momentum.

represent P by unitary operator : $(P^2 = \mathbb{1})$

$$P a_{\underline{p}}^s P = \eta_a a_{\underline{p}}^s$$
$$P b_{\underline{p}}^s P = \eta_b b_{\underline{p}}^s$$

where $\eta_a, \eta_b$ are possible phases. $PP a_{\underline{p}}^s PP = a_{\underline{p}}^s$
$\rightarrow \eta_a^2 = 1, \eta_b^2 = 1 \rightarrow \eta_{a,b} = \pm 1.$

$$P\psi(x)P = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left( u_s(p) \underbrace{P a_{\underline{p}}^s P}_{\eta_a a_{-\underline{p}}^s} e^{-ip\cdot x} + v_s(p) \underbrace{P b_{\underline{p}}^{s\dagger} P}_{\eta_b^* b_{-\underline{p}}^{s\dagger}} e^{ip\cdot x} \right)$$

Let $\tilde{p}^\mu = (E_p, -\underline{p})$ , $\tilde{x}^\mu = (t, -\underline{x})$

Then $p\cdot x = \tilde{p}\cdot\tilde{x}$

Flip $\underline{p} \to -\underline{p}$ under integral.

$$p\cdot x = \tilde{p}\cdot\tilde{x} \longrightarrow p\cdot\tilde{x}$$

$$u_s(p) \to u_s(\tilde{p}) = \begin{pmatrix} \sqrt{\sigma\cdot\tilde{p}}\,\xi_s \\ \sqrt{\bar{\sigma}\cdot\tilde{p}}\,\xi_s \end{pmatrix} = \begin{pmatrix} \sqrt{\bar{\sigma}\cdot p}\,\xi_s \\ \sqrt{\sigma\cdot p}\,\xi_s \end{pmatrix} = \gamma^0 \begin{pmatrix} \sqrt{\sigma\cdot p}\,\xi_s \\ \sqrt{\bar{\sigma}\cdot p}\,\xi_s \end{pmatrix} = \gamma^0 u_s(p)$$

$$v_s(p) \to v_s(\tilde{p}) = \begin{pmatrix} \sqrt{\sigma\cdot\tilde{p}}\,\eta_s \\ -\sqrt{\bar{\sigma}\cdot\tilde{p}}\,\eta_s \end{pmatrix} = \begin{pmatrix} \sqrt{\bar{\sigma}\cdot p}\,\eta_s \\ -\sqrt{\sigma\cdot p}\,\eta_s \end{pmatrix} = -\gamma^0 \begin{pmatrix} \sqrt{\sigma\cdot p}\,\eta_s \\ -\sqrt{\bar{\sigma}\cdot p}\,\eta_s \end{pmatrix} = -\gamma^0 v_s(p)$$

$$P\psi(x)P = \gamma^0 \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s \left(\eta_a\, u_s(p)\, a_p^s\, e^{-ip\cdot\tilde{x}} - \eta_b^*\, v_s(p)\, b_p^{s\dagger}\, e^{ip\cdot\tilde{x}}\right)$$

$$= \gamma^0\, \psi(\tilde{x})\cdot \eta_a \qquad \text{if } \eta_b^* = -\eta_a .$$

[illegible]

Free to choose arbitrary phase $\eta_a = +1$, but then must have $\underline{\eta_b = -1}$.

$\left\{ \begin{array}{l} \text{fermion-antifermion pair has intrinsic parity } -1. \\ a_p^\dagger b_p^\dagger |0\rangle \xrightarrow{P} -\, a_p^\dagger b_p^\dagger |0\rangle \end{array} \right.$

$$\boxed{\psi(t,\underline{x}) \xrightarrow{P} P\psi(t,\underline{x})P = \gamma^0\psi(t,-\underline{x})}$$

## Time-reversal

$\left.\begin{array}{l} \underline{p} \xrightarrow{T} -\underline{p} \\ \underline{J} \to -\underline{J} \end{array}\right\}$ reverses momentum & angular momentum

spinors directed along $\hat{z}$ axis $\xi_s = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$

Spinors directed along an arbitrary axis $\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

$$\xi_1 = \begin{pmatrix}\cos\theta/2 \\ e^{i\phi}\sin\theta/2\end{pmatrix} \qquad \xi_2 = \begin{pmatrix}-e^{-i\phi}\sin\frac{\theta}{2} \\ \cos\theta/2\end{pmatrix}$$

$$(\hat{n}\cdot\underline{\sigma})\,\xi_1 = +\xi_1 \;, \quad (\hat{n}\cdot\underline{\sigma})\,\xi_2 = -\xi_2$$

Time-reversal flips spin. Define "flipped" spinor

$$\xi_{-s} = -i\sigma^2 \xi_s^* = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \xi_s^*$$

So $\quad \xi_{-1} = \xi_2 \qquad \xi_{-2} = -\xi_1$

$$(\hat{n}\cdot\underline{\sigma})\,\xi_{-1} = -\,\xi_{-1}$$
$$(\hat{n}\cdot\underline{\sigma})\,\xi_{-2} = +\,\xi_2$$

Time-reversal represented by <u>anti</u> unitary operator $T$.
still satisfies $T^\dagger = T^{-1}$, but $T$ also acts on c-numbers (normal complex numbers) as

$$Tc = c^* T$$

e.g. consider an eigenstate of $H$ $|n\rangle$ with energy $E_n$ evolving in time: $e^{-iE_nt}|n\rangle$

Time-reversed state should be:

$$T\left(e^{-iE_nt}|n\rangle\right) = e^{iE_nt}\left(T|n\rangle\right)$$

i.e. evolving backward in time. So $T\underbrace{e^{-iE_nt}}_{c} = \underbrace{e^{iE_nt}}_{c^*}T$.

$T$ acts on creation/annihilation operators:

$$T a_p^s T = a_{-p}^{-s} \qquad T b_p^s T = b_p^{-s}$$

where $a_{-p}^{-s} = (a_{-p}^2, -a_{-p}^1)$, $b_{-p}^{-s} = (b_{-p}^2, -b_{-p}^1)$

additional possible phase set to 1.

Note: two successive T ~~opera~~ transformations give back original operator with minus sign:

$$T T a_{\underline{p}}^{s} T T = - a_{\underline{p}}^{s} \quad (\text{same for } b_{\underline{p}}^{s})$$

Time-reversed field:

$$T\psi(t, \underline{x})T = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s T\left(u_s(p)\, a_{\underline{p}}^{s}\, e^{-ip\cdot x} + v_s(p)\, b_{\underline{p}}^{s\dagger}\, e^{ip\cdot x}\right) T$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(u_s(p)^*\, a_{-\underline{p}}^{-s}\, e^{ip\cdot x} + v_s(p)^*\, b_{-\underline{p}}^{-s\dagger}\, e^{-ip\cdot x}\right)$$

Note: $u_s(p)^* = \begin{pmatrix} \sqrt{p\cdot\sigma^*}\, \xi_s^* \\ \sqrt{p\cdot\bar{\sigma}^*}\, \xi_s^* \end{pmatrix} = \begin{pmatrix} \sqrt{p\cdot\sigma^*}(+i\sigma^2)(-i\sigma^2)\, \xi_s^* \\ \sqrt{p\cdot\bar{\sigma}^*}(+i\sigma^2)(-i\sigma^2)\, \xi_s^* \end{pmatrix}$ $\qquad \tilde{p} = (E_p, -\underline{p})$

use: $\underline{\sigma}^* \sigma^2 = -\sigma^2 \underline{\sigma} \;\Rightarrow\; \sqrt{p\cdot\sigma^*}(+i\sigma^2) = (+i\sigma^2)\sqrt{\tilde{p}\cdot\sigma}$

$$u_s(p)^* = \underbrace{\begin{pmatrix} +i\sigma^2 & 0 \\ 0 & +i\sigma^2 \end{pmatrix}}_{\gamma^1\gamma^3} u_{-s}(\tilde{p}) = \gamma^1\gamma^3 u_{-s}(\tilde{p})$$

also $v_s(p)^* = \gamma^1\gamma^3 v_{-s}(\tilde{p})$

$$T\psi(t, \underline{x})T = \gamma^1\gamma^3 \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(u_{-s}(\tilde{p})\, a_{-\underline{p}}^{-s}\, e^{+ip\cdot x} + v_{-s}(\tilde{p})\, b_{-\underline{p}}^{-s\dagger}\, e^{-ip\cdot x}\right)$$

$\downarrow$ flip $\underline{p} \to -\underline{p}$, $s \to -s$.

$$= \gamma^1\gamma^3 \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \Big(u_s(p)\, a_{\underline{p}}^{s}\, e^{i(E_p t + \underline{p}\cdot x)} + v_s(p)\, b_{\underline{p}}^{s\dagger}\, e^{-i(E_p t + \underline{p}\cdot x)}\Big)$$

$$= \gamma^1\gamma^3 \psi(-t, \underline{x})$$

## Charge conjugation

particle $\leftrightarrow$ antiparticle

$C$ is unitary operator: $$C a_p^s C = b_p^s$$ $$C b_p^s C = a_p^s$$

Need to relate $u_s(p)$ and $v_s(p)$:

$$v_s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta_s \\ -\sqrt{p\cdot\bar\sigma}\,\eta_s \end{pmatrix} \qquad \text{take } \eta_s = \xi_{-s} = (-i\sigma^2)\xi_s^*$$

$$= \begin{pmatrix} \sqrt{p\cdot\sigma}\,(-i\sigma^2)\xi_s^* \\ -\sqrt{p\cdot\bar\sigma}\,(-i\sigma^2)\xi_s^* \end{pmatrix} = \begin{pmatrix} (-i\sigma^2)\sqrt{p\cdot\bar\sigma^*}\,\xi_s^* \\ -(-i\sigma^2)\sqrt{p\cdot\sigma^*}\,\xi_s^* \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\sigma^*}\,\xi_s^* \\ \sqrt{p\cdot\bar\sigma^*}\,\xi_s^* \end{pmatrix} = -i\gamma^2 u_s(p)^*$$

$$u_s(p) = -i\gamma^2 v_s(p)^*$$

$$C\psi(x)C = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s\left(u_s(p)\,b_p^s\,e^{-ip\cdot x} + v_s(p)\,a_p^{s\dagger}\,e^{ip\cdot x}\right)$$

$$= -i\gamma^2\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s\left(u_s(p)^*\,a_p^{s\dagger}\,e^{ip\cdot x} + v_s(p)^*\,b_p^{s}\,e^{-ip\cdot x}\right)$$

$$= -i\gamma^2\psi(x)^* = -i\gamma^2\left(\bar\psi\gamma^0\right)^T = -i\left(\bar\psi\gamma^0\gamma^2\right)^T$$

(since $\gamma^{2T} = \gamma^2$)

Dirac bilinears have well-defined transformation properties under C, P, T. $\bar{\psi}\Gamma\psi \to \begin{cases} +\bar{\psi}\Gamma\psi & \text{even} \\ -\bar{\psi}\Gamma\psi & \text{odd} \end{cases}$

e.g. consider $\bar{\psi}\psi$.

parity: $\psi(t,\underline{x}) \xrightarrow{P} \gamma^0\psi(t,-\underline{x})$

$$\bar{\psi}(t,\underline{x}) \xrightarrow{P} (\gamma^0\psi)^\dagger\gamma^0 = \psi^\dagger\gamma^0\gamma^0 = \bar{\psi}\gamma^0 = \bar{\psi}(t,-\underline{x})\gamma^0$$

so $\bar{\psi}\psi \to \bar{\psi}\gamma^0\gamma^0\psi = \bar{\psi}\psi$

is <u>even</u> under parity.

time-reversal: $\psi(t,\underline{x}) \xrightarrow{T} \gamma^1\gamma^3\psi(-t,\underline{x})$

$$\bar{\psi}(t,\underline{x}) \xrightarrow{T} (\gamma^1\gamma^3\psi)^\dagger\gamma^0 = \psi^\dagger\gamma^{3\dagger}\gamma^{1\dagger}\gamma^0 = \bar{\psi}\gamma^3\gamma^1$$

$$\bar{\psi}\psi \to \bar{\psi}\gamma^3\gamma^1\gamma^1\gamma^3\psi = \bar{\psi}\psi \quad T \text{ even}$$

charge conjugation: $\psi \xrightarrow{C} -i(\bar{\psi}\gamma^0\gamma^2)^T$

$$\bar{\psi} \xrightarrow{C} [(-i\bar{\psi}\gamma^0\gamma^2)^T]^\dagger\gamma^0$$

$$= i(\bar{\psi}\gamma^0\gamma^2)^*\gamma^0$$

$$= i(\psi^\dagger\gamma^0\gamma^0\gamma^2)^*\gamma^0$$

$$= -i\psi^T\gamma^2\gamma^0 = -i(\gamma^0\gamma^2\psi)^T$$

so $\bar{\psi}\psi \to -(\gamma^0\gamma^2\psi)^T(\bar{\psi}\gamma^0\gamma^2)^T$

$$= -\gamma^0_{ab}\gamma^2_{bc}\psi_c\,\bar{\psi}_d\gamma^0_{de}\gamma^2_{ea}$$

fields anticommute

$$= +\bar{\psi}_d\gamma^0_{de}\gamma^2_{ea}\gamma^0_{ab}\gamma^2_{bc}\psi_c = \bar{\psi}\gamma^0\gamma^2\gamma^0\gamma^2\psi = \bar{\psi}\psi \quad C \text{ even}$$