Quantum electrodynamics

Theory of Dirac fermion ψ (electron) and massless spin-1 field A_μ (photon)

$$\mathcal{L}_{QED} = \underbrace{\bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi}_{\mathcal{L}_{Dirac}} \underbrace{- \frac{1}{4} F_{\mu\nu}F^{\mu\nu}}_{\mathcal{L}_{Maxwell}} + \underbrace{\mathcal{L}_{int}}_{\text{interaction.}}$$

Interaction is determined by symmetry.

Recall: $\mathcal{L}_{Dirac}$ invariant under $\psi \to e^{i\alpha}\psi$.

<u>Global</u> phase rotation since α is constant.

<u>Local</u> phase rotation: $\psi(x) \longrightarrow e^{i\alpha(x)}\psi(x)$

Rotation parameter $\alpha(x)$ depends on spacetime

Different rotation at different space time points.

$$\mathcal{L}_{Dirac} \longrightarrow \bar{\psi} e^{-i\alpha}(i\gamma^\mu \partial_\mu - m) e^{i\alpha}\psi = \mathcal{L}_{Dirac} - \bar{\psi}\gamma^\mu\psi\, \partial_\mu \alpha$$

$\mathcal{L}_{Dirac}$ is not invariant by itself. To cancel extra term, promote

$$\partial_\mu \to D_\mu = \partial_\mu + ieA_\mu \qquad \text{covariant derivative}$$

with A_μ transforming as

$$A_\mu \to A_\mu - \frac{1}{e}\partial_\mu \alpha$$

$$\bar{\Psi} i\gamma^\mu D_\mu \Psi \to \bar{\Psi} i\gamma^\mu \partial_\mu \Psi - \cancel{\bar{\Psi}\gamma^\mu \Psi \partial_\mu \alpha}$$

$$- e\bar{\Psi}\gamma^\mu A_\mu \Psi + \cancel{\bar{\Psi}\gamma^\mu \Psi \partial_\mu \alpha}$$

$$= \bar{\Psi} i\gamma^\mu D_\mu \Psi$$

is invariant under local phase rotations.

Dirac fermion + $\underbrace{\text{local symmetry}}_{\text{(gauge symmetry)}}$ $\Rightarrow$ introduce $\underbrace{\text{vector field } A_\mu}_{\text{(gauge field)}}$

$$\mathcal{L}_{Dirac} = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi \implies \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi$$

If A_μ is a dynamical field (satisfies Maxwell's eqns)
then introduce $\mathcal{L}_{maxwell} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$

$$\mathcal{L}_{QED} = \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

so $\mathcal{L}_{int} = \bar{\Psi} i\gamma^\mu (ie)A_\mu \Psi = -e\underbrace{\bar{\Psi}\gamma^\mu\Psi}_{j^\mu}A_\mu$ current

$F_{\mu\nu} \to F_{\mu\nu}$ under $A_\mu \to A_\mu - \frac{1}{e}\partial_\mu \alpha$

So $\mathcal{L}_{maxwell}$ is also invariant.

e is the electric charge (of the proton).

Generalized QED with N Dirac fermions ψ_j $(j=1,\dots,N)$

$$\mathcal{L}_{Dirac} = \sum_{j=1}^{N} \bar{\psi}_j \left(i\gamma^\mu \partial_\mu - m_j \right) \psi_j$$

Different fermions can have different electric charge $q_j e$.

- electron (e), muon (μ), tau (τ) : $q_{e,\mu,\tau} = -1$.
- up (u), charm (c), top (t) quarks : $q_{u,c,t} = +2/3$
- down (d), strange (s), bottom (b) quarks : $q_{d,s,b} = -1/3$.

Transform as: $\psi_j(x) \longrightarrow e^{-iq_j \alpha(x)} \psi_j(x)$

covariant derivative : $D_\mu \psi_j = (\partial_\mu - ieq_j A_\mu)\psi_j$

$$\mathcal{L}_{int} = \underbrace{\sum_{j=1}^{N} eq_j \bar{\psi}_j \gamma^\mu \psi_j}_{\substack{\text{total current } j^\mu \text{ summing} \\ \text{over all charged species.}}} A_\mu$$

Feynman rules for QED

Dirac propagator: $\left(\text{analog of } D(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle \right)$

$$\langle 0| \psi_a(x) \bar{\psi}_b(y) |0\rangle$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \sum_{s,s'} \langle 0| u_s(p)_a \, a_p^s \, e^{-ip\cdot x}$$

$$\times \bar{u}_{s'}(q)_b \, a_q^{s'\dagger} e^{iq\cdot y} |0\rangle$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \underbrace{\sum_s u_s(p)_a \bar{u}_s(p)_b}_{\gamma^\mu p_\mu + m} e^{-ip\cdot(x-y)}$$

Feynman slash notation: $\not{p} = \gamma^\mu p_\mu$

$$\not{\partial} = \gamma^\mu \partial_\mu$$

$$\langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}(\not{p}+m)_{ab}\, e^{-ip\cdot(x-y)}$$

$$= (i\not{\partial}_x + m)_{ab}\, D(x-y)$$

$$\langle 0|\bar{\psi}_b(y)\psi_a(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\underbrace{\sum_s v_s(p)_a \bar{v}_s(p)_b}_{(\not{p}-m)_{ab}}\, e^{-ip\cdot(y-x)}$$

$$= -(i\not{\partial}_x + m)_{ab}\, D(y-x)$$

Feynman propagator:

$$S_F(x-y)_{ab} = \langle 0|T\,\psi_a(x)\bar{\psi}_b(y)|0\rangle$$

$$= \begin{cases} \langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle & x^0 > y^0 \\ -\langle 0|\bar{\psi}_b(y)\psi_a(x)|0\rangle & y^0 > x^0 \end{cases}$$

$$= (i\not{\partial}_x + m)_{ab} \begin{cases} D(x-y) & x^0 > y^0 \\ D(y-x) & y^0 > x^0 \end{cases}$$

$$= (i\not{\partial}_x + m)_{ab}\, D_F(x-y)$$

$$= (i\not{\partial}_x + m)_{ab} \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\varepsilon}\, e^{-ip\cdot(x-y)}$$

$$= \int \frac{d^4p}{(2\pi)^4}\frac{i(\not{p}+m)}{p^2-m^2+i\varepsilon}\, e^{-ip\cdot(x-y)}$$

Wick's theorem for fermions

Recall: for interacting scalar fields $\phi_I(x_i) = \phi_i$

$$T(\phi_1 \phi_2 \cdots \phi_n) = N(\phi_1 \phi_2 \cdots \phi_n + \text{all possible contractions})$$

$$T(\phi_1 \phi_2 \phi_3 \phi_4) = N(\phi_1\phi_2\phi_3\phi_4 + \overbrace{\phi_1\phi_2}\phi_3\phi_4 + \cdots + \overbrace{\phi_1\phi_2}\,\overbrace{\phi_3\phi_4} + \cdots)$$

$$\langle 0 | T(\phi_1 \cdots \phi_n) | 0 \rangle = \text{all possible contractions where all fields are contracted}$$

$$= D_F(x_1 - x_2) \cdots D_F(x_{n-1} - x_n) + \text{permutations.}$$

Normal ordering defined by writing $\phi_I = \phi_I^+ + \phi_I^-$,
where

$$\phi_I^\pm(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \begin{cases} a_p\, e^{-ip\cdot x} & (+) \\ a_p^\dagger\, e^{+ip\cdot x} & (-) \end{cases},$$

and commuting all ϕ_I^+ fields to the right of the ϕ_I^- fields. Each commutation picks up a contraction $\overbrace{\phi\phi} = D_F$.

For fermions, rules are similar:

- divide $\psi_I = \psi_I^+ + \psi_I^-$, $\bar\psi_I = \bar\psi_I^+ + \bar\psi_I^-$

where

$$\psi_I^\pm = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \begin{cases} a_p\, e^{-ip\cdot x}\, u_s(p) & (+) \\ b_p^\dagger\, e^{ip\cdot x}\, v_s(p) & (-) \end{cases}$$

$$\bar\psi_I^\pm = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \begin{cases} b_p\, e^{-ip\cdot x}\, \bar v_s(p) & (+) \\ a_p^\dagger\, e^{ip\cdot x}\, u_s(p) & (-) \end{cases}$$

- anti commute ψ_I^+ and $\bar{\psi}_I^+$ to the right of ψ_I^-, $\bar{\psi}_I^-$ fields.

- each anticommutation picks up a contraction

$$\psi \bar{\psi} = S_F .$$

Wick's theorem for fermions: denote $\psi_i = \psi_I(x_i)$, $\bar{\psi}_j = \bar{\psi}_I(x_j)$

$$T(\psi_1 \bar{\psi}_2 \psi_3 \cdots) = N(\psi_1 \bar{\psi}_2 \psi_3 \cdots + \text{all possible contractions})$$

same as for scalar fields.
but need to keep track of extra minus signs.

e.g. $\psi_1 \psi_2 \bar{\psi}_3 \bar{\psi}_4 = - \psi_1 \bar{\psi}_3 \psi_2 \bar{\psi}_4$

$$= - S_F(x_1 - x_3)\, \psi_2 \bar{\psi}_4$$

$$\bar{\psi}_1 \psi_2 \bar{\psi}_3 \psi_4 = (-1)^3\, \psi_4 \bar{\psi}_1\, \psi_2 \bar{\psi}_3$$

$$= - S_F(x_4 - x_1)\, S_F(x_2 - x_3)$$

always anticommute all contractions to be in the form $\psi_i \bar{\psi}_j = S_F(x_i - x_j)$

e.g. $\psi_1 \cdots \psi_i \cdots \bar{\psi}_j \cdots = \pm S_F(x_i - x_j)\, \psi_1 \cdots$

$\pm$ depending on even (+) or odd (−) number of anticommutations to put $\psi_i \bar{\psi}_j$ next to each other (in order)

Yukawa theory

Simplified model for QED

Dirac fermion ψ + real scalar ϕ

$$\mathcal{L} = \mathcal{L}_{Dirac} + \mathcal{L}_{KG} + \mathcal{L}_{int}$$

$$= \underbrace{\bar{\psi}(i\not{\partial} - m)\psi}_{Dirac} + \underbrace{\tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}m_\phi^2\phi^2}_{KG} + \underbrace{g\bar{\psi}\psi\phi}_{\mathcal{L}_{int}}$$

Yukawa interaction.

Consider 2→2 scattering of ψ particles:

$$\psi(\underline{p},s)\,\psi(\underline{k},r) \to \psi(\underline{p}',s')\,\psi(\underline{k}',r')$$

Want to compute:

$$\langle \underline{p}',s';\underline{k}',r' |\, T\exp\left(i\int d^4z\, \mathcal{L}_I(z)\right) | \underline{p},s;\underline{k},r\rangle$$

Leading contribution to matrix element $\mathcal{M}$ at $\mathcal{O}(g^2)$

$$\langle \underline{p}',s';\underline{k}',r' |\, T\left(-\tfrac{1}{2}\right)g^2\int d^4z\int d^4w\; \bar{\psi}_z\psi_z\phi_z\,\bar{\psi}_w\psi_w\phi_w\, | \underline{p},s;\underline{k},r\rangle$$

Use Wick's theorem to evaluate T-ordered product.

Also need contraction with external states:

$$\psi_I(x)\,|\underline{p},s\rangle = \int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_q}}\sum_r\left(u_r(q)\,a_q^r e^{-iq\cdot x} + v_r(q)\,b_q^{r\dagger}e^{iq\cdot x}\right)\sqrt{2E_p}\,a_p^{s\dagger}|0\rangle$$

$$= u_s(p)\,e^{-ip\cdot x}\,|0\rangle$$

$$\langle p,s|\bar\psi_I(x) = \langle 0|\,\bar u_s(p)\, e^{ip\cdot x}$$

$$\bar\psi_I(x)\,|p,s\rangle = \langle p,s|\,\psi_I(x) = 0$$

Similarly for antiparticle states: $|\widetilde{p,s}\rangle = \sqrt{2E_p}\, b_p^{s\dagger}|0\rangle$

$$\bar\psi_I(x)\,|\widetilde{p,s}\rangle = \bar v_s(p)\, e^{-ip\cdot x}$$

$$\langle\widetilde{p,s}|\,\psi_I(x) = v_s(p)\, e^{ip\cdot x}$$

$$\psi_I(x)\,|\widetilde{p,s}\rangle = \langle\widetilde{p,s}|\,\bar\psi_I(x) = 0.$$

So we have:

$$\langle p',s';k',r'|\left(-\tfrac{1}{2}g^2\right)\int d^4z\int d^4w\; \bar\psi_z\psi_z\phi_z\,\bar\psi_w\psi_w\phi_w\,|p,s;k,r\rangle$$

$$= -\frac{g^2}{2}\int d^4z\int d^4w\,\langle p',s';k',r'|\,\bar\psi_{za}\bar\psi_{wb}\psi_{wb}\psi_{za}\phi_z\phi_w\,|p,s;k,r\rangle\,(-1)^2$$

$$= -\frac{g^2}{2}\int d^4z\int d^4w\;\bar u_{s'}(p')_a\,\bar u_{r'}(k')_b\,u_r(k)_b\,u_s(p)_a$$

$$\times\, D_F(z-w)\, e^{i(p'\cdot z+k'\cdot w-p\cdot z-k\cdot w)}$$

$$= -\frac{g^2}{2}\int d^4z\int d^4w\;\bar u_{s'}(p')\,u_s(p)\;\bar u_{r'}(k')\,u_r(k)$$

$$\times\int\frac{d^4q}{(2\pi)^4}\,\frac{i}{q^2-m_\phi^2+i\varepsilon}\;e^{-iq\cdot(z-w)}\;e^{i(p'\cdot z+k'\cdot w-p\cdot z-k\cdot w)}$$

$$= -\frac{g^2}{2}\int\frac{d^4q}{(2\pi)^4}\,\frac{i}{q^2-m_\phi^2+i\varepsilon}\;(2\pi)^4\delta^4(p-p'+q)\;(2\pi)^4\delta^4(k-k'-q)$$

$$\times\;\bar u_{s'}(p')\,u_s(p)\;\bar u_{r'}(k')\,u_r(k)$$

$$= -\frac{g^2}{2} \frac{i}{(p-p')^2 - m_\phi^2 + i\varepsilon} \bar{u}_{s'}(p') u_s(p) \quad \bar{u}_{r'}(k') u_r(k)$$

$$\times (2\pi)^4 \delta^4 (p+k-p'-k')$$

Another possible contraction:

$$\langle \underline{p}', s'; \underline{k}', r' | \left(-\tfrac{1}{2} g^2\right) \int d^4z \int d^4w \; \bar{\Psi}_z \Psi_z \phi_z \bar{\Psi}_w \Psi_w \phi_w \; | \underline{p}, s ; \underline{k}, r \rangle$$

$$= -\frac{1}{2} g^2 \int d^4z \int d^4w \langle \underline{p}', s'; \underline{k}', r' | \bar{\Psi}_{za} \bar{\Psi}_{wb} \; \Psi_{za} \Psi_{wb} \; \phi_z \phi_w \; | \underline{p}, s; \underline{k}, r \rangle (-1)$$

$$= \cancel{-}\frac{1}{2} g^2 \int d^4z \int d^4w \;\; \bar{u}_s(p')_a \, \bar{u}_{r'}(k')_b \;\; u_s(p)_b \;\; u_r(k)_a \quad (-1)$$

$$\times D_F(z-w) \; e^{i(p'\cdot z + k' \cdot w - p \cdot w - k \cdot z)}$$

$$= + \frac{g^2}{2} \frac{i}{(p-k')^2 - m_\phi^2 + i\varepsilon} \bar{u}_{s'}(p') u_r(k) \quad \bar{u}_{r'}(k') u_s(p)$$

$$\times (2\pi)^4 \delta^4 (p+k-p'-k')$$

Two more contractions obtained by flipping $w \leftrightarrow z$.
$\Rightarrow$ multiply each by 2.

$$\sum \text{all possible contractions} = i\mathcal{M} \times (2\pi)^4 \delta^4 (p+k-p'-k')$$

Matrix element is

$$i\mathcal{M}\left(\psi(p,s)\psi(k,r) \to \psi(p',s')\psi(k',r')\right)$$

$$= -ig^2 \left(\frac{\bar{u}_{s'}(p') u_s(p) \;\; \bar{u}_{r'}(k') u_r(k)}{(p-p')^2 - m_\phi^2 + i\varepsilon} - \frac{\bar{u}_{s'}(p') u_r(k) \, \bar{u}_{r'}(k') u_s(p)}{(p-k')^2 - m_\phi^2 + i\varepsilon} \right)$$

Feynman rules for Yukawa theory (momentum space)

① external states:

external fermion line $\quad p,s \rightarrow \; = u(p,s)$ (incoming)

$\rightarrow p,s = \bar{u}(p,s)$ (outgoing)

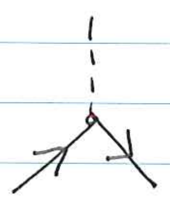
external antifermion line $\quad \tilde{p},s \xleftarrow{p \rightarrow} \; = \bar{v}(p,s)$ (incoming)

$\xleftarrow[p \rightarrow]{} \tilde{p},s = v(p,s)$ (outgoing)

Note: arrow on fermion line represents direction of "fermion number" (i.e. charge). Opposite to direction of momentum for antifermions.

external scalar line $\quad q - - - - \; = 1$ (incoming or outgoing)

② Vertices

 $= ig$

③ Impose momentum conservation at each vertex.

④ Propagators (internal lines)

Scalar propagator $\quad \overset{q \rightarrow}{- - - - - -} = \dfrac{i}{q^2 - m_\phi^2 + i\varepsilon}$

fermion propagator $\quad \overset{q \rightarrow}{\longrightarrow} = \dfrac{i(\not{q} + m)}{q^2 - m^2 + i\varepsilon}$

⑤ Integrate over all undetermined momenta (only for diagrams with loops)

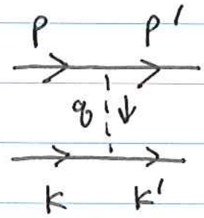
⑥ determine relative minus sign between diagrams (over all sign doesn't matter for $|\mathcal{M}|^2$)

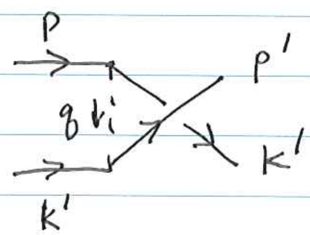
⑦ Each closed fermion loop gets factor of (-1)

Note 1: No symmetry factor. No degeneracy in contracting fields differently with $\mathcal{L}_{int}$ since $\bar{\Psi}, \Psi, \phi$ all contract differently.

Factor of $\frac{1}{n!}$ from $\exp\left(i \int d^4z\, \mathcal{L}_I(z)\right)$ cancels with $n!$ from exchanging spacetime points (e.g. $z \leftrightarrow w$)

Note 2: Dirac spinor indices $(a, b, \ldots)$ are contracted along the fermion line.

$i\mathcal{M} =$ 

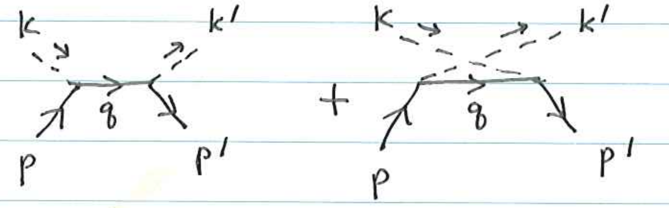
$+$ 

(spin indices omitted)

$$= (ig)^2 \Big[\bar{u}(p') u(p)\ \bar{u}(k') u(k)\ \frac{i}{(p-p')^2 - m_\phi^2 + i\varepsilon}$$

$$- \bar{u}(k') u(p)\ \bar{u}(p') u(k)\ \frac{i}{(p-k')^2 - m_\phi^2 + i\varepsilon} \Big]$$

example: $\psi(p)\,\phi(k) \to \psi(p')\,\phi(k')$

$i\mathcal{M} =$ 

$q = p + k$ $\qquad\qquad$ $q = p - k'$

First term:

$$\langle p', s'; \underline{k}'|\ \frac{(ig)^2}{2}\int d^4z \int d^4w\ \bar{\psi}_z \psi_z \phi_z\ \bar{\psi}_w \psi_w \phi_w\ |p, s; \underline{k}\rangle \times 2$$

$$\psi_z \bar{\psi}_w = S_F(z-w) = \int \frac{d^4q}{(2\pi)^4} \frac{i(\not{q}+m)}{q^2 - m^2 + i\varepsilon} e^{-iq\cdot(z-w)}$$

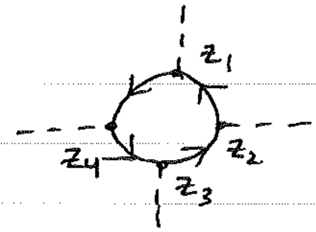
2nd term:

$$\langle p', s'; \underline{k}'|\ \frac{(ig)^2}{2}\int d^4z \int d^4w\ \bar{\psi}_z \psi_z \phi_z\ \bar{\psi}_w \psi_w \phi_w\ |p, s; \underline{k}\rangle$$

no relative minus sign since ϕ field commutes.

$$i\mathcal{M} = (ig)^2 \Big[\bar{u}(p') \frac{i(\not{p}+\not{k}+m)}{(p+k)^2 - m^2 + i\varepsilon} u(p)$$

$$+ \bar{u}(p') \frac{i(\not{p}-\not{k}'+m)}{(p-k')^2 - m^2 + i\varepsilon} u(p) \Big]$$

Closed femion loop: e.g. subdiagram 

fermion contractions are

$$\bar\psi_{1a}\psi_{1a}\,\bar\psi_{2b}\psi_{2b}\,\bar\psi_{3c}\psi_{3c}\,\bar\psi_{4d}\psi_{4d}$$ anti-commute ψ_{4d} through

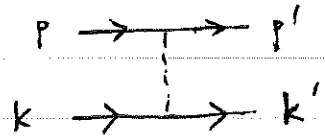
$$= (-1)\, S_F(x_1-x_2)_{ab}\, S_F(x_2-x_3)_{bc}\, S_F(x_3-x_4)_{cd}\, S_F(x_4-x_1)_{da}$$

$$= (-1)\, \mathrm{Tr}\left[S_F(x_1-x_2)\, S_F(x_2-x_3)\, S_F(x_3-x_4)\, S_F(x_4-x_1)\right]$$

closed fermion loop always gets factor of (-1) and involves a Trace.

Yukawa potential

Matrix element for $\psi(p)\,\psi(k) \to \psi(p')\,\psi(k')$ related to QM potential in nonrelativistic limit.

$i\mathcal{M} =$ 

$$= \frac{-ig^2}{(p-p')^2 - m_\phi^2}\; \bar u_{s'}(p')\, u_s(p)\;\; \bar u_{r'}(k')\, u_r(k)$$

Let $q = p' - p$. and $E_p \simeq E_{p'} \simeq m$ in N.R. limit.

also $u = \sqrt{m}\begin{pmatrix}\xi_s \\ \xi_s\end{pmatrix}$.

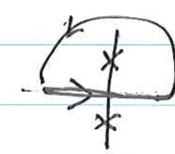
So $$i\mathcal{M} = \frac{ig^2}{|q|^2 + m_\phi^2} \times (2m)^2\, \xi_{s'}^\dagger \xi_s\; \xi_{r'}^\dagger \xi_r$$

Coefficient $\dfrac{g^2}{|\underset{\sim}{q}|^2 + m_\phi^2} = -\tilde{V}(\underset{\sim}{q})$ is the momentumspace representation of the potential (with a minus sign).

$$V(\underline{r}) = \int \frac{d^3\underset{\sim}{q}}{(2\pi)^3}\, e^{+i\underset{\sim}{q}\cdot\underline{r}}\, \tilde{V}(\underset{\sim}{q}) = \int \frac{d^3\underset{\sim}{q}}{(2\pi)^3} \frac{-g^2}{|\underset{\sim}{q}|^2 + m_\phi^2}\, e^{i\underset{\sim}{q}\cdot\underline{r}}$$

$$= \frac{-g^2}{(2\pi)^2} \int_0^\infty q^2\, dq \int_{-1}^{+1} d\cos\vartheta\, \frac{e^{iqr\cos\vartheta}}{q^2 + m_\phi^2}$$

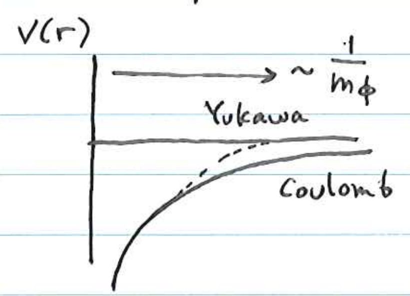
$$= -\frac{g^2}{(2\pi)^2} \int_0^\infty \frac{q^2\, dq}{q^2 + m_\phi^2} \cdot \frac{1}{iqr}\left(e^{iqr} - e^{-iqr}\right)$$



$$= -\frac{g^2}{4\pi^2 r i} \int_{-\infty}^{\infty} \frac{q\, dq\, e^{iqr}}{(q + im_\phi)(q - im_\phi)} =$$

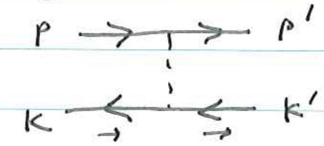
$$= -\frac{g^2}{4\pi^2 r i} \cdot 2\pi i\, \frac{im_\phi}{2im_\phi}\, e^{-m_\phi r} = -\frac{g^2}{4\pi r}\, e^{-m_\phi r}$$

Yukawa potential (screened Coulomb potential)



Next, consider particle-antiparticle scattering:

$$\Psi(\underline{p})\,\bar\Psi(\underline{k}) \to \Psi(\underline{p}')\,\bar\Psi(\underline{k}')$$



Similar to previous case:

$$\bar u_{r'}(\underline{k}')\,u_r(\underline{k}) \to \bar v_r(\underline{k})\,v_{r'}(\underline{k}') = m\,(\eta^\dagger_{r'}, -\eta^\dagger_{r})\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\eta_{r'}\\-\eta_{r'}\end{pmatrix}$$
$$= 2m^2\,\xi^\dagger_{r'}\xi_r \qquad\qquad = -2m^2\,\eta^\dagger_r\eta_r$$

also relative (-1) from fermion contractions.

Fermion contraction for particle-particle scattering

$$\langle \underline{p}', \underline{k}' |\, \bar\Psi\Psi \;\; \bar\Psi\Psi \,| \underline{p}, \underline{k}\rangle = \langle 0 |\, a_{\underline{k}'} a_{\underline{p}'}\; \bar\Psi\Psi \;\; \bar\Psi\Psi \; a^\dagger_{\underline{p}} a^\dagger_{\underline{k}} \,|0\rangle$$
$$= \langle 0|\, a_{k'} a_{p'}\; \bar\Psi\;\bar\Psi \;\; \Psi\;\Psi\; a^\dagger_{p} a^\dagger_{\underline{k}}\,|0\rangle \times \underbrace{(-1)^2}_{+1}$$

Fermion contraction for particle-antiparticle scattering

$$\langle \underline{p}', \tilde{\underline{k}}' |\, \bar\Psi\Psi \;\; \bar\Psi\Psi \,| \underline{p}, \tilde{\underline{k}}\rangle = \langle 0 |\, b_{\underline{k}'} a_{\underline{p}'}\; \bar\Psi\Psi \;\; \bar\Psi\Psi \; a^\dagger_{\underline{p}} b^\dagger_{\underline{k}} \,|0\rangle$$
$$= \langle 0|\, b_{\underline{k}'} a_{p'}\; \bar\Psi\Psi \;\; \bar\Psi\Psi\; a^\dagger_{p}\, b^\dagger_{k}\,|0\rangle\, (-1)^3$$
$$\downarrow$$
$$-1$$

So particle-antiparticle scattering has the same amplitude

$\Rightarrow$ same potential $V(r) = -\dfrac{g^2}{4\pi r}\, e^{-m_\phi r}$

Also same for antiparticle-antiparticle scattering.

Yukawa interaction is universally attractive between all combinations of particles and antiparticles.

Also follows from $\bar{\Psi}\Psi \xrightarrow{C} \bar{\Psi}\Psi$ even under charge conjugation.

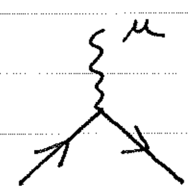
<u>Feynman rules for QED</u> (Ψ = electron)

$$\mathcal{L} = \bar{\Psi}(i\not{D} - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad D_\mu = \partial_\mu + ieA_\mu$$

$$\rightarrow \mathcal{L}_{int} = -e\bar{\Psi}\gamma^\mu\Psi A_\mu$$

Fermion external states & propagators $\rightarrow$ same as before

Vertex



$= -ie\gamma^\mu$

Feynman rules for photons:

(Free) Maxwell's equations $\partial_\mu F^{\mu\nu} = 0$

$$\rightarrow \partial^2 A^\nu - \partial^\nu \partial_\mu A^\mu = 0$$

Gauge field A^μ contains a redundancy in its degree of freedom $\rightarrow$ "fix" the gauge by imposing additional <u>gauge-fixing condition</u>.

Convenient choice: Lorentz gauge $\partial_\mu A^\mu = 0$

$\Rightarrow$ Equation of motion becomes $\partial^2 A^\mu = 0$

A^μ satisfies massless KG eqn.

Mode expansion:

$$A^\mu(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=0}^{3} \left(a_p^s \, \varepsilon_s^\mu(p) \, e^{-ip\cdot x} + a_p^{s\dagger} \, \varepsilon_s^\mu(p)^* e^{ip\cdot x} \right)$$

where $\varepsilon_s^\mu(p)$ are 4 polarization vectors.

Note: only the two transverse polarizations are physical

e.g. for $\vec{p} = (0,0,p)$, only

$\varepsilon_1^\mu = (0,1,0,0)$ and $\varepsilon_2^\mu = (0,0,1,0)$

(or linear combinations) are physical.

also have longitudinal polarization

e.g. $\varepsilon_3^\mu = (0,0,0,1)$

and "time-like" polarization

e.g. $\varepsilon_0^\mu \propto (1,0,0,0)$

But convenient to keep extra polarizations + rule

$$\sum_{s=0}^{3} \varepsilon_s^\mu \varepsilon_s^{\nu *} = -g^{\mu\nu}$$

It turns out that summing over 0 & 3 polarizations cancel out

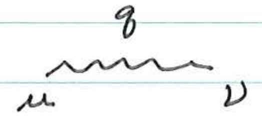
Photon propagator:

$$\langle 0|T A^\mu(x) A^\nu(y)|0\rangle = -g^{\mu\nu} D_F(x-y)$$

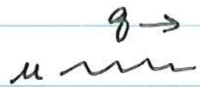
using $[a_p^s, a_q^{r\dagger}] = (2\pi)^3 \delta^3(p-q)\,\delta_{rs}$

and $\sum_s \varepsilon_s^\mu(p)\,\varepsilon_s^\nu(p)^* = -g^{\mu\nu}$

Feynman rule:

photon propagator 

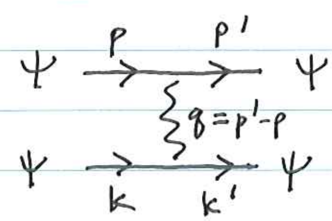
$= \dfrac{-ig_{\mu\nu}}{q^2+i\varepsilon}$

external photon 

$= \varepsilon^\mu(q)$ (incoming)

$= \varepsilon^\mu(q)^*$ (outgoing)

<u>Coulomb potential</u> particle-particle scattering



$$= \bar u_{s'}(p')(-ie\gamma^\mu)u_s(p)\frac{-ig_{\mu\nu}}{(p-p')^2+i\varepsilon}\bar u_{r'}(k')(-ie\gamma^\nu)u_r(k)$$

$$= +\frac{ie^2}{q^2+i\varepsilon}\,\bar u(p')\gamma^\mu u_s(p)\,\bar u(k')\gamma_\mu u(k)$$

Nonrelativistic limit:

$\mu=0$ term:

$$\bar u(p')\gamma^0 u_s(p) = (\sqrt m)^2 (\xi_{s'}^\dagger, \xi_{s'}^\dagger)\gamma^0\gamma^0\begin{pmatrix}\xi_s\\ \xi_s\end{pmatrix} = 2m\,\xi_{s'}^\dagger\xi_s$$

$\mu = i = 1,2,3$ term

$$\bar{u}(p')\gamma^i u_s(p) = (\sqrt{m})^2 \left(\xi_{s'}^\dagger, \xi_{s'}^\dagger\right)\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} \xi_s \\ \xi_s \end{pmatrix}$$

$$= m\left(\xi_{s'}^\dagger \sigma^i \xi_s - \xi_{s'}^\dagger \sigma^i \xi_s\right) = 0$$

at $\mathbf{p}, \mathbf{p}' \to 0$.

Also $q^2 \cong -|\mathbf{q}|^2$.

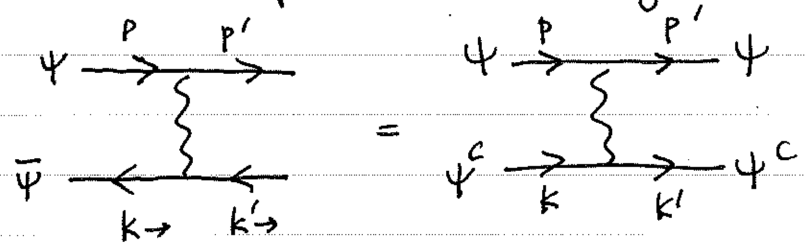
So we have $i\mathcal{M} = \underbrace{\dfrac{-ie^2}{|\mathbf{q}|^2}} (2m)^2 \xi_{s'}^\dagger \xi_s \, \xi_{r'}^\dagger \xi_r$

$$\Rightarrow \tilde{V}(\mathbf{q}) = \frac{e^2}{|\mathbf{q}|^2}$$

Fourier transform: $V(r) = \dfrac{e^2}{4\pi r}$

repulsive coulomb potential.

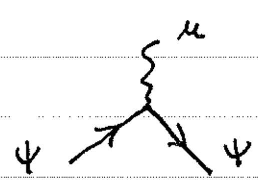
particle - antiparticle scattering:



How does $\bar{\Psi}\gamma^\mu\Psi$ transform under C?

$$\bar{\Psi}\gamma^\mu\Psi \xrightarrow{C} \overline{\Psi}^c\gamma^\mu\Psi^c = -\bar{\Psi}\gamma^\mu\Psi$$

So $\mathcal{L}_{int} = -e\bar{\Psi}\gamma^\mu\Psi A_\mu = +e\bar{\Psi}^c\gamma^\mu\Psi^c A_\mu$



$= -ie\gamma^\mu$

$= +ie\gamma^\mu$

So $\tilde{V}(q) = -\dfrac{e^2}{|\vec{q}|^2} \Rightarrow V(r) = -\dfrac{e^2}{4\pi r}$

particle and antiparticle attract.

<u>Elementary processes in QED</u> (PS ch. 5)

example: $e^+e^- \to \mu^+\mu^-$

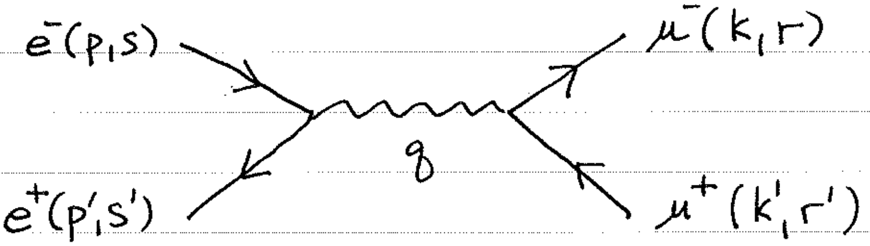
$$\mathcal{L} = \bar{e}(i\not{D} - m_e)e + \bar{\mu}(i\not{D} - m_\mu)\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

e, μ Dirac fermions with charge $-e$

$m_e \approx .511$ MeV

$m_\mu \approx 105.6$ MeV

$i\mathcal{M} =$



$$q = p + p' = k + k'$$

$$= \bar{v}_s(p')(-ie\gamma^\mu)u(p,s)\,\frac{-ig_{\mu\nu}}{q^2+i\varepsilon}\,\bar{u}_r(k)(-ie\gamma^\nu)v_{r'}(k')$$

$$= \frac{ie^2}{q^2}\,\bar{v}(p')\gamma^\mu u(p)\;\bar{u}(k)\gamma_\mu v(k')$$ (spin indices suppressed)

We are going to compute the <u>unpolarized</u> cross section:

- initial state is average of all possible spins
- final state is summed over all possible spins

$$\sigma(e^+e^- \to \mu^+\mu^-) = \frac{1}{4E_pE_{p'}v_{rel}} \int \frac{d^3k}{(2\pi)^3 2E_k} \int \frac{d^3k'}{(2\pi)^3 2E_{k'}} (2\pi)^4 \delta^4(p+p'-k-k')$$

$$\times \underbrace{\frac{1}{2}\sum_s \frac{1}{2}\sum_{s'}}_{\text{average initial spins}} \underbrace{\sum_r \sum_{r'}}_{\text{sum final spins}} |\mathcal{M}|^2$$

First, compute $|\mathcal{M}|^2 = \mathcal{M}^*\mathcal{M}$

$$\mathcal{M}^* = \frac{e^2}{q^2}\left(\bar{v}(p')\gamma^\nu u(p)\right)^*\left(\bar{u}(k)\gamma_\nu v(k')\right)^*$$

note: $$\left(\bar{u}(k)\gamma_\nu v(k')\right)^* = v(k')^\dagger \gamma_\nu^\dagger \gamma^{0\dagger} u(k)$$

$$= \bar{v}(k') \underbrace{\gamma^0 \gamma_\nu^\dagger \gamma^0}_{=\gamma_\nu} u(k) = \bar{v}(k')\gamma_\nu u(k)$$

So $$\mathcal{M}^* = \frac{e^2}{q^2}\bar{u}(p)\gamma^\nu v(p')\,\bar{v}(k')\gamma_\nu u(k)$$

Then $$|\mathcal{M}|^2 = \frac{e^4}{q^4}\left(\bar{u}(p)\gamma^\nu v(p')\,\bar{v}(p')\gamma^\mu u(p)\right)$$

$$\times\left(\bar{v}(k')\gamma_\nu u(k)\,\bar{u}(k)\gamma_\mu v(k')\right)$$

Next, we use sum over spins to evaluate the spinor terms.

e.g. $$\sum_{s,s'} \bar{u}(p,s)\gamma^{\nu} v(p',s')\,\bar{v}(p',s')\gamma^{\mu} u(p,s)$$

$$= \sum_{s,s'} \bar{u}(p,s)_a\, \gamma^{\nu}_{ab}\, v(p',s')_b\, \bar{v}(p',s')_c\, \gamma^{\mu}_{cd}\, u(p,s)_d$$

Spin sums: $$\sum_s u(p,s)_d\, \bar{u}(p,s)_a = (\not{p}+m)_{da}$$

$$\sum_{s'} v(p',s')_b\, \bar{v}(p',s')_c = (\not{p}'-m)_{bc}$$

So we have:

$$(\not{p}+m_e)_{da}\, \gamma^{\nu}_{ab}\, (\not{p}'-m_e)_{bc}\, \gamma^{\mu}_{cd}$$

$$= \mathrm{Tr}\left[(\not{p}+m_e)\gamma^{\nu}(\not{p}'-m_e)\gamma^{\mu}\right]$$

Sums over spins $\rightarrow$ traces over Dirac matrices.

$$\sum_{s,s',r,r'} |\mathcal{M}|^2 = \frac{e^4}{q^4}\,\mathrm{Tr}\left[(\not{p}+m_e)\gamma^{\nu}(\not{p}'-m_e)\gamma^{\mu}\right]$$

$$\times\, \mathrm{Tr}\left[(\not{k}+m_{\mu})\gamma_{\mu}(\not{k}'-m_{\mu})\gamma_{\nu}\right]$$

Traces of Dirac matrices

Need to evaluate trace of n Dirac matrices:

$n=0$: $\mathrm{Tr}[\mathbb{1}] = 4$

$n=1$: $\mathrm{Tr}[\gamma^\mu] = \mathrm{Tr}\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} = 0$

$n=2$: $\mathrm{Tr}[\gamma^\mu\gamma^\nu] = \frac{1}{2}\mathrm{Tr}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) = \frac{1}{2}\mathrm{Tr}\{\gamma^\mu,\gamma^\nu\}$

$$= \frac{1}{2}\mathrm{Tr}[2\,\mathbb{1}\,g^{\mu\nu}] = 4g^{\mu\nu}$$

$n=$ odd:

$$\mathrm{Tr}[\gamma^{\mu_1}\cdots\gamma^{\mu_n}] = \mathrm{Tr}[\gamma^{\mu_1}\cdots\gamma^{\mu_n}\gamma^5\gamma^5]$$
$$= (-1)^n\,\mathrm{Tr}[\gamma^5\gamma^{\mu_1}\cdots\gamma^{\mu_n}\gamma^5]$$
$$= (-1)^n\,\mathrm{Tr}[\gamma^{\mu_1}\cdots\gamma^{\mu_n}] \quad = 0$$

Trace of odd number of γ^μ vanishes.

$n=4$: $\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = \mathrm{Tr}[-\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho + 2\gamma^\mu\gamma^\nu g^{\rho\sigma}]$

$$= \mathrm{Tr}[\gamma^\mu\gamma^\sigma\gamma^\nu\gamma^\rho - 2\gamma^\mu g^{\nu\sigma}\gamma^\rho + 2\gamma^\mu\gamma^\nu g^{\rho\sigma}]$$

$$= \mathrm{Tr}[-\gamma^\sigma\gamma^\mu\gamma^\nu\gamma^\rho + 2g^{\mu\sigma}\gamma^\nu\gamma^\rho - 2\gamma^\mu\gamma^\rho g^{\nu\sigma} + 2\gamma^\mu\gamma^\nu g^{\rho\sigma}]$$

$$= -\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] + 8g^{\mu\sigma}g^{\nu\rho} - 8g^{\mu\rho}g^{\nu\sigma} + 8g^{\mu\nu}g^{\rho\sigma}$$

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$$

Also need traces involving $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

$$\mathrm{Tr}[\gamma^5] = i\,\mathrm{Tr}[\gamma^0\gamma^1\gamma^2\gamma^3] = 0$$

$\mathrm{Tr}[\gamma^\mu\gamma^5] = 0$ (odd number of γ matrices)

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^5] = 0$$

$$\text{e.g. } \mathrm{Tr}[\gamma^0\gamma^1\gamma^5] = i\,\mathrm{Tr}[\gamma^0\gamma^1\gamma^0\gamma^1\gamma^2\gamma^3]$$
$$= i\,\mathrm{Tr}[\gamma^2\gamma^3] = 0$$

$\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^5] = 0$ (odd number of γ matrices)

$\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\kappa\gamma^5] = 0$ unless $\mu, \nu, \lambda, \kappa$ all different

$$\text{eg. } \mathrm{Tr}[\gamma^0\gamma^1\gamma^2\gamma^3\gamma^5] = i\,\mathrm{Tr}[\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3]$$
$$= -i\,\mathrm{Tr}[\mathbb{1}] = -4i$$

$$\text{So } \mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\kappa\gamma^5] = -4i\,\varepsilon^{\mu\nu\lambda\kappa}$$

Lastly, when two γ-matrices are contracted together, can remove them.

$$\gamma^\mu\gamma_\mu = g_{\mu\nu}\gamma^\mu\gamma^\nu = \tfrac{1}{2}g_{\mu\nu}\{\gamma^\mu, \gamma^\nu\} = \tfrac{1}{2}g_{\mu\nu}\,2g^{\mu\nu} = 4$$

$$\mathrm{Tr}[(\not p + m_e)\gamma^\nu(\not p' - m_e)\gamma^\mu]$$

$$= \mathrm{Tr}[\not p \gamma^\nu \not p' \gamma^\mu] - \mathrm{Tr}[\gamma^\nu \gamma^\mu] m_e^2$$

$$= 4(p^\nu p'^\mu + p^\mu p'^\nu - p\cdot p' g^{\mu\nu}) - 4m_e^2 g^{\mu\nu}$$

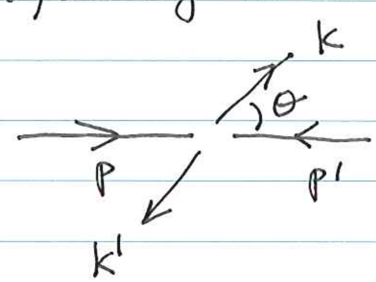
$$\mathrm{Tr}[(\not k + m_\mu)\gamma_\mu(\not k' - m_\mu)\gamma_\nu]$$

$$= 4(k_\mu k'_\nu + k_\nu k'_\mu - k\cdot k' g_{\mu\nu}) - 4m_\mu^2 g_{\mu\nu}$$

$$\Sigma|\mathcal{M}|^2 = \frac{32e^4}{q^4}\left((p\cdot k)(p'\cdot k') + (p\cdot k')(p'\cdot k) + m_\mu^2(p\cdot p')\right)$$

neglecting me ($m_e = 0$)

Compute σ by working in CM frame:



$p = (E, 0, 0, E)$

$p' = (E, 0, 0, -E)$

$k = (E, \underline{k})$

$k' = (E, -\underline{k})$

$|\underline{k}| = \sqrt{E^2 - m_\mu^2}$

$$p\cdot k = p'\cdot k' = E^2 - E|\underline{k}|\cos\theta \qquad q^2 = 4E^2$$

$$p\cdot k' = p'\cdot k = E^2 + E|\underline{k}|\cos\theta \qquad p\cdot p' = 2E^2$$

$$\Sigma|\mathcal{M}|^2 = 4e^4\left[\left(1 + \frac{m_\mu^2}{E^2}\right) + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]$$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_{cm}^2}\sqrt{1 - \frac{m_\mu^2}{E^2}}\left[\left(1 + \frac{m_\mu^2}{E^2}\right) + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]$$

$$\sum |\mathcal{M}|^2 = \frac{32e^4}{(4E^2)^2}\left((E^2 - E|\underline{k}|\cos\theta)^2 + (E^2 + E|\underline{k}|\cos\theta)^2 + 2E^2 m_\mu^2\right)$$

$$= \frac{2e^4}{E^4}\left(2E^4 + 2E^2(E^2 - m_\mu^2)\cos^2\theta + 2E^2 m_\mu^2\right)$$

$$= 4e^4\left(\left(1 + \frac{m_\mu^2}{E^2}\right) + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2\theta\right)$$

$$\sigma = \frac{1}{4E^2 v_{rel}} \int \frac{d^3k}{(2\pi)^3}\frac{1}{2E_k}\int \frac{d^3k'}{(2\pi)^3}\frac{1}{2E_{k'}} (2\pi)^4 \delta^4(p+p'-k-k') \frac{1}{4}\sum|\mathcal{M}|^2$$

In relativistic limit: $v_{rel} = 2$.

Phase space integral:

$$I_2 = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2E_k}\int\frac{d^3k'}{(2\pi)^3}\frac{1}{2E_{k'}}(2\pi)^4\delta^4(p+p'-k-k')$$

$$= \int\frac{d^3k}{(2\pi)^3}\frac{1}{4E^2}(2\pi)\,\delta(2E - 2E_k)$$

$$= \frac{1}{(2\pi)^2}\frac{1}{4E^2}(2\pi)\int_0^\infty dE_k E_k\, k \underset{\int d\cos\theta}{\wedge} \frac{1}{2}\delta(E_k - E) = \frac{1}{16\pi}\frac{|\underline{k}|}{E}\int d\cos\theta$$

$$= \frac{1}{32\pi^2}\frac{|\underline{k}|}{E}\int d\Omega = \frac{1}{32\pi^2}\sqrt{1 - m_\mu^2/E^2}\int d\Omega$$

$$\sigma = \frac{1}{8E^2}\frac{1}{32\pi^2}\sqrt{1-m_\mu^2/E^2}\int d\Omega\; e^4\left(\left(1+\frac{m_\mu^2}{E^2}\right) + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2\theta\right)$$

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{256\pi^2 E^2}\left(1 - m_\mu^2/E^2\right)^{1/2}\left(\left(1 + \frac{m_\mu^2}{E^2}\right) + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2\theta\right)$$

define $\alpha = e^2/4\pi$, $E_{cm} = 2E$ total CM energy

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_{cm}^2}\left(1 - m_\mu^2/4E_{cm}^2\right)^{1/2}\left(\left(1 + \frac{m_\mu^2}{4E_{cm}^2}\right) + \left(1 - \frac{m_\mu^2}{4E_{cm}^2}\right)\cos^2\theta\right)$$

total cross section:

$$\int d\Omega = 4\pi$$

$$\int d\Omega \cos^2\theta = 2\pi \int_{-1}^{+1} d\cos\theta \cos^2\theta = \frac{4\pi}{3}$$

$$\sigma = \frac{\pi\alpha^2}{E_{cm}^2}\left(1 - m_\mu^2/4E_{cm}^2\right)^{1/2}\left(1 + \frac{m_\mu^2}{4E_{cm}^2} + \frac{1}{3} - \frac{1}{3}\frac{m_\mu^2}{4E_{cm}^2}\right)$$

$$= \frac{4\pi\alpha^2}{3E_{cm}^2}\underbrace{\left(1 - \frac{m_\mu^2}{4E_{cm}^2}\right)^{1/2}}\left(1 + \frac{m_\mu^2}{2E_{cm}^2}\right)$$

phase space factor $\sim |k|/E$

σ vanishes for $E \to m_\mu$

Relativistic limit: $E_{cm} \gg m_\mu \;\rightarrow\; \sigma = \frac{4\pi\alpha^2}{3E_{cm}^2}$

σ had dim. of $[\text{area}] = \frac{1}{[\text{mass}]^2}$

Must scale as $\sigma \propto 1/E_{cm}^2$ since E_{cm} is only dimensionful parameter